ON CONNECTED IRRESOLVABLE HAUSDORFF SPACES[1]

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Hewitt [2, p. 327] has raised, and Padmavally [4] has answered to the affirmative, the question of the existence of connected irresolvable Hausdorff spaces. The purpose of this note is to prove a general existence theorem for connected irresolvable Hausdorff spaces which shows that the class of such spaces is more numerous than previously supposed.

In the discussion preceding Theorem 2 the underlying point set of a topological space will be denoted by X. A topology on X will be denoted by either R or T. When it is necessary to distinguish between different topologies on the same set X, subscripts will be used.

We recall some definitions and notations. If R_1 and R_2 are two topologies for X, R_2 is an *expansion* of R_1 if $R_1 \subseteq R_2$. A topology R for X is *irresolvable* if in the space (X, R) there is no dense set D for which $X-D$ is also dense. The *dispersion character* of a topology R for X, denoted by $\Delta(R)$, is the least cardinality of a nonempty open set in R.

Throughout this paper we shall be concerned with a special class of expansions of a given topology on X. We make the following

DEFINITION. Let (X, R) be a topological space, $\{D_\alpha : \alpha \in A\}$ be the set of all R-dense subsets of X. An expansion R^* of R is *admissible* if R^* has a subbasis of the form $R \cup \{D_\beta : \beta \in B \subseteq A\}$ and if each of the sets D_β is R^*-dense.

We note that any admissible expansion R^* of R has a unique subbasis $S^* = R \cup \{D_\beta : \beta \in B \subseteq A\}$ with B maximal. $S^* = R \cup \{D : D$ is R^*-dense and R^*-open$\}$ is such a subbasis for R^*. This subbasis will be called the *admissible subbasis* for R^* and the subset $B \subseteq A$ will be called the *index class corresponding to* R^*.

LEMMA 1. *Let R^* be an admissible expansion of the connected topology R. Then R^* is connected.*

PROOF. Suppose R^* is not connected. Then there exist nonempty disjoint open sets O_1 and O_2 in R^* such that $O_1 \cup O_2 = X$. Since R^* is admissible, we can write

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$$O_1 = \bigcup_{\phi \in P} \left(O_\phi \cap \left(\bigcap_{i=1}^{n} D_{\phi_i} \right) \right), \quad O_2 = \bigcup_{\psi \in Q} \left(O_\psi \cap \left(\bigcap_{j=1}^{m} D_{\psi_j} \right) \right),$$

where O_ϕ, $O_\psi \in R$ and P, Q are index sets. But then

$$\varnothing = O_1 \cap O_2 = \bigcup_{\phi,\psi} \left(O_\phi \cap O_\psi \cap \left(\bigcap_{i=1}^{n} D_{\phi_i} \right) \cap \left(\bigcap_{j=1}^{m} D_{\psi_j} \right) \right).$$

Hence, for all ϕ, ψ we have

$$O_\phi \cap O_\psi \cap \left(\bigcap_{i=1}^{n} D_{\phi_i} \right) \cap \left(\bigcap_{j=1}^{m} D_{\psi_j} \right) = \varnothing.$$

Now since R^* is admissible

$$O_\phi \cap O_\psi \cap \left(\bigcap_{i=1}^{n} D_{\phi_i} \right) \cap \left(\bigcap_{j=1}^{m-1} D_{\psi_j} \right) = O^*$$

is R^*-open and D_{ψ_m} is R^*-dense. Since $O^* \cap D_{\psi_m} = \varnothing$, we have $O^* = \varnothing$. Therefore

$$O_\phi \cap O_\psi \cap \left(\bigcap_{i=1}^{n} D_{\phi_i} \right) \cap \left(\bigcap_{j=1}^{m-1} D_{\psi_j} \right) = \varnothing.$$

Proceeding by induction, it follows that $O_\phi \cap O_\psi = \varnothing$ for all ϕ and ψ. But then $\bigcup_{\phi \in P} O_\phi$ and $\bigcup_{\psi \in Q} O_\psi$ are nonempty disjoint R-open sets which cover X. Hence R is not connected, which is a contradiction.

REMARK. Let X be a set of infinite cardinality τ_0. Let τ be an infinite cardinal number such that $\tau_0 \geqq \tau$. Let $\mathfrak{F}(\tau)$ be the set of all subsets of X whose complement has cardinality $< \tau$. Then $\mathfrak{F}(\tau)$ is a filter.

PROOF. This is obvious.

DEFINITION. Let X and τ be as in the above remark and let R be a topology on X. Then R_τ shall denote the topology on X with subbasis $S_\tau = R \cup \mathfrak{F}(\tau)$. Since $\mathfrak{F}(\tau)$ is a filter, a basis for R_τ may be taken to be the family of all sets of the form $O \cap F$ where $O \in R$ and $F \in \mathfrak{F}(\tau)$.

LEMMA 2. *Let R be a connected topology with $\Delta(R) \geqq \tau$. Then R_τ is connected and $\Delta(R_\tau) \geqq \tau$.*

PROOF. Since $\Delta(R) \geqq \tau$, every set $F \in \mathfrak{F}(\tau)$ is R-dense, for no R-open set can be contained in $X - F$. Since it is clear that every $F \in \mathfrak{F}(\tau)$ is R_τ-dense, it follows directly from the definition of R_τ that R_τ is an admissible expansion of R. Hence R_τ is connected by Lemma 1.

If $\Delta(R_\tau) < \tau$, then there is a nonempty R_τ-open set O whose cardinality is less than τ. Hence $X - O \in \mathfrak{F}(\tau)$ and, consequently, $X - O$ is also R_τ-open. Thus R_τ is not connected, which is a contradiction.

We now proceed to the main theorem.

THEOREM 1. *Let τ be an infinite cardinal number. Let R be a connected topology for X with $\Delta(R) \geqq \tau$. Then there exists a connected irresolvable expansion R^* of R with $\Delta(R^*) \geqq \tau$.*

PROOF. By Lemma 2, R_τ is a connected expansion of R with $\Delta(R_\tau) \geqq \tau$. We shall expand R_τ to obtain R^*.

Let $\mathfrak{A}$ be the set of all admissible expansions T of R_τ with the following ordering: $T_1 \leqq T_2$ if $S_1 \subseteq S_2$, where S_i is the admissible subbasis for T_i, $i=1, 2$. We note that $T_1 < T_2$ if and only if there is some set D which is T_2-dense and T_2-open and which is also T_1-dense, but is not T_1-open.

$\mathfrak{A}$ is not empty for R_τ belongs to $\mathfrak{A}$. Let $\mathfrak{L} = \{T_\gamma : \gamma \in C\}$ be a linearly ordered subfamily of $\mathfrak{A}$ and $B_\gamma \subseteq A$ be the index class corresponding to T_γ. Consider the topology T^* with subbasis $S^* = R_\tau \cup \{D_\delta : \delta \in \bigcup_{\gamma \in C} B_\gamma \subseteq A\}$. T^* is an admissible expansion of R_τ, for let D be any of the dense sets D_δ and let O be any nonempty open set in the basis for T^* generated by S^*. Then $O = O_1 \cap (\bigcap_{i=1}^n D_{\beta_i})$, where $O_1 \in R_\tau$. Since there are only a finite number of the sets D, D_{β_i}, $i=1, \cdots, n$, and since $\mathfrak{L}$ is linearly ordered, there exists an index γ_0 such that D, $D_{\beta_i} \in S_{\gamma_0}$ for $i=1, \cdots, n$. But T_{γ_0} is admissible; hence O is T_{γ_0}-open and $O \cap D \neq \varnothing$. But O was an arbitrary element of a basis for T^*. Hence D is T^*-dense and T^* is admissible. Thus every linearly ordered subfamily of $\mathfrak{A}$ has an upper bound in $\mathfrak{A}$; hence, by Zorn's Lemma, $\mathfrak{A}$ has a maximal element R^* with admissible subbasis $S^* = R_\tau \cup \{D_\beta : \beta \in B^*\}$.

R^* is irresolvable, for if it is not, there is an R^*-dense set D such that $X - D$ is also R^*-dense. Hence D is not R^*-open. But D is dense in R_τ and the topology T with subbasis $S^* \cup \{D\}$ is an admissible expansion of R_τ strictly greater than R^*. This contradicts the maximality of R^*.

Since R^* is an admissible expansion of R_τ and R_τ is connected, R^* is connected. Since R^* is a connected expansion of R_τ, $\Delta(R^*) \geqq \tau$, by the same argument used in Lemma 2 to show $\Delta(R_\tau) \geqq \tau$. Clearly R^* is an expansion of R.

Hence R^* is the desired topology.

COROLLARY. *Let R be a connected Hausdorff topology for X with $\Delta(R) \geqq \tau$, where τ is an infinite cardinal number. Then there exists a connected irresolvable Hausdorff expansion T of R with $\Delta(T) \geqq \tau$.*

PROOF. Since the property of being Hausdorff is invariant under expansions, we may apply Theorem 1 to R and let T be the topology R^*.

As a consequence of Theorem 1, to establish the existence of connected irresolvable Hausdorff spaces with arbitrary infinite dispersion character, it suffices to establish the existence of connected Hausdorff spaces of arbitrary infinite dispersion character. The existence of the latter spaces is established by

THEOREM 2. *Let τ be an infinite cardinal number. Then there exists a connected Hausdorff space* (X, R) *with* $\Delta(R)=\tau$.

PROOF. For $\tau=\aleph_0$, let (X, R) be any countable connected Hausdorff space. (Examples of such spaces are well known. See [5] or [1].) Then $\Delta(R)=\aleph_0$, for, if not, there is a nonempty finite open set O. But since R is Hausdorff, the finite set O is closed. Thus since O is both open and closed in R, R is not connected, which is a contradiction.

For $\tau>\aleph_0$, let C be a countable connected Hausdorff space, let T be a set of cardinality τ, and let $p: T\rightarrow C$ be any function. Then the set X of all functions $q: T\rightarrow C$ which agree with p at all but a finite number of points with the topology R of pointwise convergence (see [3, p. 217]) is the desired space (X, R). For clearly R is connected, Hausdorff, and $\Delta(R)\geqq\tau$. But also each function in X is determined by its behavior on a finite subset F of T and for each such subset F, there are only $\aleph_0^{\text{card } F}=\aleph_0$ possible functions. Since card $T=\tau$, there are τ finite subsets of T. Hence there are $\aleph_0\cdot\tau=\tau$ points in X. Thus $\Delta(R)\leqq\tau$ and the last assertion of the theorem is established.

Combining the corollary above with Theorem 2 we obtain

COROLLARY. *Let τ be an infinite cardinal number. Then there exists a connected irresolvable Hausdorff space* (X, R) *with* $\Delta(R)=\tau$.

If the above results are applied to the real line, we see that there is a connected topology finer than the usual topology with the property that each nonempty open set has $\mathfrak{c}$ points in which U and the complement of U are never simultaneously dense.

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REFERENCES

1. R. H. Bing, *A connected countable Hausdorff space*, Proc. Amer. Math. Soc. **4** (1953), 474.

2. E. Hewitt, *A problem of set theoretic topology*, Duke Math. J. **10** (1943), 309–333.

3. J. L. Kelley, *General topology*, Van Nostrand, Princeton, N. J., 1955.

4. K. Padmavally, *An example of a connected irresolvable Hausdorff space*, Duke Math. J. **20** (1953), 513–520.

5. P. Urysohn, *Über die Mächtigkeit der zusammenhängenden Menge*, Math. Ann. **94** (1925), 262–295.

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