# ON THE EXPRESSION OF A NUMBER AS THE SUM OF TWO SQUARES IN TOTALLY REAL ALGEBRAIC NUMBER FIELDS[1]

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**Introduction.** Let $K$ be a totally real algebraic number field of degree $n$ and with discriminant $d$. Let $\mathfrak{a}$ be an ideal of $K$ which may be integral or fractional. The number of solutions of the equation

$$\xi = \mu^2 + \nu^2 \qquad (\xi \in \mathfrak{a})$$

in numbers $\mu, \nu \in \mathfrak{a}$ is denoted by $f(\xi, \mathfrak{a})$. For $x_1, \cdots, x_n$ being positive real numbers the following theorem will be proved:

THEOREM.

$$\sum_{0<\xi^{(h)}<x_h;\,\mathfrak{a}\mid\xi} f(\xi, \mathfrak{a}) = \frac{\pi^n}{dN\mathfrak{a}^2}(x_1 \cdots x_n) + R(x_1, \cdots, x_n).$$

*(The index $h$ always takes on the values $1, \cdots, n$ if not otherwise indicated.) For any $\delta > 0$, $x_1 \cdots x_n \to \infty$, then*

$$R(x_1, \cdots, x_n) = O((x_1 \cdots x_n)^{n/(n+1)+\delta})$$

*holds.*

This result has been already proved in [4] for the case $n = 2$, $\mathfrak{a} = (1)$. There was also shown that

$$\limsup_{x_1x_2\to\infty} \frac{R(x_1, x_2)}{(x_1x_2)^{1/4}} > 0.$$

For the proof of the theorem an identity given by Siegel in [5] for real quadratic number fields is generalized to totally real algebraic number fields. This identity will be applied to the problem in a similar way as it was done in [4].

1. In what follows the real numbers $c_1, \cdots, c_5$ are constants greater than 1 which only depend on the field $K$ and the ideal $\mathfrak{a}$ if not otherwise indicated. We define $S(\alpha) = \alpha^{(1)} + \cdots + \alpha^{(n)}$, $N(\alpha) = \alpha^{(1)} \cdots \alpha^{(n)}$ for numbers $\alpha \in K$. Let $r = n - 1$, and let

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[2] We introduce Hecke's characters for a number $\alpha\in K$ with respect to these unit $\eta_1, \cdots, \eta_r$.

$$m = 2\pi i \begin{pmatrix} m_1 \\ \cdot \\ \cdot \\ \cdot \\ m_r \end{pmatrix}$$

where $m_1, \cdots, m_r$ are rational integers.

The set of squares of all units of $K$ forms a group $G$ which may be generated by the $r$ independent units $\eta_1, \cdots, \eta_r$.[2] For this purpose let $E$ be the $r \times n$ matrix $(e_p^{(q)})$, $q = 1, \cdots, r$; $p = 1, \cdots, n$ (see [2]), and let

$$a = \begin{pmatrix} \log|\alpha^{(1)}| \\ \cdot \\ \cdot \\ \cdot \\ \log|\alpha^{(n)}| \end{pmatrix}.$$

Then following Hecke's definition we set

$$\lambda_m(\alpha) = \exp\{m^T E a\}. \tag{1}$$

If $\eta \in G$ it follows from the definition of the numbers $e_p^{(q)}$ that

$$\lambda_m(\alpha\eta) = \lambda_m(\alpha).$$

Two numbers $\alpha, \beta \neq 0$, 0 of $K$ are called "associated" if their quotient is an element of the group $G$. Otherwise $\alpha, \beta$ are called "not associated."

LEMMA 1. *If $x$ is a positive real number then*

$$\sum_{N(\xi) \leq x}' f(\xi, \mathfrak{a}) = O(x)$$

*where the dash at the sign of summation indicates that the sum is to be taken over a set of not associated numbers $\xi \in \mathfrak{a}$.*

PROOF. For every number $\alpha$ of $K$ there exists a number $c_1$ and a unit $\eta \in G$ which only depends on $\alpha$ such that the following $n$ inequalities hold:

$$c_1^{-1} |N(\alpha)|^{1/n} \leqq |\alpha^{(h)} \eta^{(h)}| \leqq c_1 |N(\alpha)|^{1/n}, \qquad h = 1, \cdots, n,$$

(see [6, Hilfssatz 6]). Because of

$$f(\eta\xi, \mathfrak{a}) = f(\xi, \mathfrak{a}), \qquad \eta \in G \tag{2}$$

we may choose the set of not associated numbers $\xi$ such that the following inequalities are satisfied:

$$c_1^{-1}(N\xi)^{1/n} \leqq \xi^{(h)} \leqq c_1(N\xi)^{1/n}, \qquad h = 1, \cdots, n.$$

Whence we have

$$\sum_{N(\xi)\leq x}{}' f(\xi, \mathfrak{a}) \leqq \sum_{0<\xi^{(h)}<c_1x^{1/n};\mathfrak{a}|\xi} f(\xi, \mathfrak{a}).$$

Since $f(\xi, \mathfrak{a})$ is the number of distinct pairs $(\mu, \nu)$, $\mu, \nu \in \mathfrak{a}$ with $\xi = \mu^2 + \nu^2$ it is sufficient to estimate the number of elements $\mu \in \mathfrak{a}$ which satisfy the inequalities $|\mu^{(h)}| < c_2 x^{1/2n}$, $h = 1, \cdots, n$. Let $\alpha_1, \cdots, \alpha_n$ be a basis of the ideal $\mathfrak{a}$. We have to estimate the number of distinct $n$-tuples of rational integers $(k_1, \cdots, k_n)$ for which the inequalities

$$-c_2 x^{1/2n} < \sum_{p=1}^{n} k_p \alpha_p^{(h)} < c_2 x^{1/2n}, \qquad h = 1, \cdots, n$$

hold. Since $|\det(\alpha_p^{(h)})| = N\mathfrak{a}\sqrt{d} \neq 0$ we obtain that there are at most $c_3\sqrt{x}$ of such $n$-tuples. This proves the lemma.

For each character (1) we define the function

$$\Phi_m(s, \mathfrak{a}) = \sum_{\xi}{}' \frac{f(\xi, \mathfrak{a})\lambda_m(\xi)}{N(\xi)^s},$$

where by $s = \sigma + it$ a complex variable is denoted. Applying the method of partial summation it is an easy consequence of Lemma 1 that the functions $\Phi_m(s, \mathfrak{a})$ converge absolutely and uniformly for $\sigma > 1$.

Let $R$ be the determinant

$$\begin{vmatrix} 1 & \log \eta_1^{(1)} & \cdots & \log \eta_r^{(1)} \\ \vdots & \vdots & & \vdots \\ 1 & \log \eta_1^{(n)} & \cdots & \log \eta_r^{(n)} \end{vmatrix};$$

moreover, we introduce the abbreviation

$$E_p(m) = 2\pi \sum_{q=1}^{r} m_q e_p^{(q)}, \qquad p = 1, \cdots, n.$$

Then the following lemma holds:

LEMMA 2. *Let $x_1, \cdots, x_n$ be positive real numbers and let*

$$g(x_1, \cdots, x_n) = \sum_{0<\xi^{(h)}x_h<1;\mathfrak{a}|\xi} f(\xi, \mathfrak{a}) \prod_{p=1}^{n} (1 - \xi^{(p)}x_p).$$

*Then we have for $\sigma > 1$:*

$$g(x_1, \cdots x_n) = \frac{n}{2\pi i\,|R|} \sum_{m_1,\cdots,m_r;=-\infty}^{+\infty} \int_{\sigma-i\infty}^{\sigma+i\infty} \Phi_m(s, \mathfrak{a})$$

$$\cdot \prod_{p=1}^{n} \frac{x_p^{-s+iE_p(m)}}{(s - iE_p(m))(s + 1 - iE_p(m))}\, ds.$$

Proof. The proof of the given identity proceeds on the same lines as the proof in the case $n=2$ given in [5]. We define the column vectors

$$k = \begin{pmatrix} k_1 \\ \cdot \\ \cdot \\ \cdot \\ k_r \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ \cdot \\ \cdot \\ \cdot \\ v_r \end{pmatrix}, \quad y^{(p)} = \begin{pmatrix} \log \eta_1^{(p)} \\ \cdot \\ \cdot \\ \cdot \\ \log \eta_r^{(p)} \end{pmatrix} \qquad (p = 1, \cdots, n),$$

where $k_1, \cdots, k_r$ are rational integers and $v_1, \cdots, v_r$ are real variables. Making the substitution

$$x_p = u \exp\{v^T y^{(p)}\}, \qquad p = 1, \cdots, n \tag{3}$$

we observe that the function $g(x_1, \cdots, x_n)$ becomes a periodic function with respect to $v_1, \cdots, v_r$ because of property (2). The period is 1 with respect to each of the variables. Furthermore, $g(x_1, \cdots, x_n)$ is a continuous function and has piecewise continuous partial derivatives with respect to $v_1, \cdots, v_r$. Whence $g(x_1, \cdots, x_n)$ furnishes an absolutely convergent Fourier series. Denoting the right-hand side of (3) by $t^{(p)}(v)$ its coefficient is given by:

$$a_m(u) = \int_0^1 \cdots \int_0^1 \exp\{-v^T m\} \sum_{0<\xi^{(h)} t^{(h)}(v)<1;\, \mathfrak{a}|\xi} f(\xi, \mathfrak{a})$$

$$\cdot \prod_{p=1}^{n} (1 - \xi^{(p)} t^{(p)}(v))\, dv_1 \cdots dv_r$$

$$= \int_0^1 \cdots \int_0^1 \exp\{-v^T m\} \sum_{0<N(\xi)<u^{-n}}' f(\xi, \mathfrak{a})$$

$$\cdot \sum_{k_1,\cdots,k_r=-\infty}^{\infty} \sum_{0<\xi^{(h)} t^{(h)}(v+k)<1} \prod_{p=1}^{n} (1 - \xi^{(p)} t^{(p)}(v + k))\, dv_1 \cdots dv_r$$

$$= \int_0^1 \cdots \int_0^1 \exp\{-v^T m\} \sum_{k_1,\cdots,k_r-\infty}^{\infty} \sum_{0<N(\xi)<u^{-n}}' f(\xi, \mathfrak{a})$$

$$\cdot \sum_{0<\xi^{(h)} t^{(h)}(v+k)<1} \prod_{p=1}^{n} ( \cdots )\, dv_1 \cdots dv_r.$$

We are allowed to interchange the integration and the summation with respect to $k_1, \cdots, k_r$ because the sum is finite. Making the change of variables $v_q+k_q \to v_q$, $q=1, \cdots, r$, we obtain:

$$a_m(u) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp\{-v^T m\} \sum_{0<\xi^{(h)}t^{(h)}(v)<1}' f(\xi, \mathfrak{a}) \cdot \prod_{p=1}^{n} (1-\xi^{(p)}t^{(p)}(v))dv_1 \cdots dv_r.$$

Now we form the integral $\int_0^\infty u^{ns-1} a_m(u)du$ for $\sigma>1$. Making the change of variables (3) we get:

$$\int_0^\infty u^{ns-1}a_m(u)du$$

$$= \frac{1}{|R|}\int_0^\infty \cdots \int_0^\infty \prod_{p=1}^{n} x_p^{s-1-iE_p(m)} \sum_{0<\xi^{(h)}x_h<1}' f(\xi, \mathfrak{a}) \cdot \prod_{p=1}^{n}(1-\xi^{(p)}x_p)dx_1 \cdots dx_n$$

$$= \frac{1}{|R|}\sum_{\xi}' f(\xi, \mathfrak{a}) \prod_{p=1}^{n} \int_0^{(\xi^{(p)})^{-1}} x_p^{s-1-iE_p(m)}(1-\xi^{(p)}x_p)dx_p$$

$$= \frac{1}{|R|}\Phi_m(s, \mathfrak{a}) \prod_{p=1}^{n} [(s-iE_p(m))(s+1-iE_p(m))]^{-1}.$$

The application of Mellin's inversion formula yields for $\sigma>1$:

$$a_m(u) = \frac{n}{2\pi i|R|}\int_{\sigma-i\infty}^{\sigma+i\infty} u^{-ns}\Phi_m(s, \mathfrak{a}) \prod_{p=1}^{n} [(s-iE_p(m))(s+1-iE_p(m))]^{-1}ds.$$

Since

$$g(x_1, \cdots, x_n) = \sum_{m_1, \cdots, m_r=-\infty}^{\infty} a_m(u) \exp\{v^T m\},$$

this proves the lemma.

Let

$$F(v_1, \cdots, v_n) = \sum_{0<\xi^{(h)}<v_h} f(\xi, \mathfrak{a}).$$

Then we have

$$
\begin{aligned}
(x_1 \cdots x_n) g\left(\frac{1}{x_1}, \cdots, \frac{1}{x_n}\right) &= \sum_{0<\xi^{(h)}<x_h;\, \mathfrak{a} \mid \xi} f(\xi, \mathfrak{a}) \prod_{p=1}^{n} (x_p - \xi^{(p)}) \\
&= \sum_{0<\xi^{(h)}<x_h;\, \mathfrak{a} \mid \xi} \int_{\xi^{(1)}}^{x_1} \cdots \int_{\xi^{n}}^{x_n} f(\xi, \mathfrak{a}) dv_1 \cdots dv_n \\
&= \int_0^{x_1} \cdots \int_0^{x_n} F(v_1, \cdots, v_n) dv_1 \cdots dv_n.
\end{aligned}
$$

An elementary calculation furnishes the result:

$$
\begin{aligned}
&\int_0^{y_1} \cdots \int_0^{y_n} F(x_1 + v_1, \cdots, x_n + v_n) dv_1 \cdots dv_n \\
(4) \qquad &= \frac{n}{2\pi i \mid R \mid} \sum_{m_1, \cdots, m_r = -\infty}^{+\infty} \int_{\sigma - i\infty; \sigma > 1}^{\sigma + i\infty} \\
&\cdot \prod_{p=1}^{n} \frac{(y_p + x_p)^{s+1-iE_p(m)} - x_p^{s+1-iE_p(m)}}{(s - iE_p(m))(s + 1 - iE_p(m))} \Phi_m(s, \mathfrak{a}) ds.
\end{aligned}
$$

2. The left-hand side of (4) may be abbreviated by $J$. Since $f(\xi, \mathfrak{a}) \geqq 0$ we obtain the inequality:

$$
F(x_1, \cdots, x_n) \leqq (y_1 \cdots y_n)^{-1} J \leqq F(x_1 + y_1, \cdots, x_n + y_n).
$$

We observe from this inequality that the asymptotic behaviours of $F(x_1, \cdots, x_n)$ and $(y_1 \cdots y_n)^{-1}J$ are the same. Therefore we shall try to find an approximation of $J$. For this purpose the functions $\Phi_m(s, \mathfrak{a})$ are analytically continued over the whole $s$-plane. Let:

$$
\Theta(z_1, \cdots, z_n; \mathfrak{a}) = \sum_{\mathfrak{a} \mid \mu} \exp\left\{ - \frac{\pi}{\sqrt[n]{(dN\mathfrak{a}^2)}} \sum_{p=1}^{n} \mu^{(p)2} z_p \right\},
$$

$z_1, \cdots, z_n$ being complex variables with $\operatorname{Re} z_h > 0$, $h = 1, \cdots, n$; then Hecke proved in [3]:

$$
(5) \qquad \Theta(z_1, \cdots, z_n; \mathfrak{a}) = (z_1 \cdots z_n)^{-1/2} \Theta\left(\frac{1}{z_1}, \cdots, \frac{1}{z_n}; \frac{1}{\mathfrak{a}\mathfrak{d}}\right),
$$

where $\mathfrak{d}$ is the ramification ideal of the field $K$. Well known calculations and the application of (5) lead to the equation:

$$\left(\frac{dN\mathfrak{a}^2}{\pi^n}\right)^s \frac{1}{|R|}\Phi_m(s,\mathfrak{a})\prod_{p=1}^{n}\Gamma(s-iE_p(m))$$

$$= \frac{b_m}{s(s-1)} + \int_{u=1}^{u=\infty}\int_{-1/2}^{1/2}\cdots\int_{-1/2}^{1/2}\Big[\Theta^2\big(u\eta_1^{(1)v_1}\cdots\eta_r^{(1)v_r},\cdots,$$

$$\cdot u\eta_1^{(n)v_1}\cdots\eta_r^{(n)v_r};\mathfrak{a}\big)-1\Big]u^{ns}\exp\{-v^Tm\}dv_1\cdots dv_r\frac{du}{u} \tag{6}$$

$$+\int_{u=1}^{u=\infty}\int_{-1/2}^{1/2}\cdots\int_{-1/2}^{1/2}\Big[\Theta^2\Big(u\eta_1^{(1)v_1}\cdots\eta_r^{(1)v_r},\cdots,$$

$$u\eta_1^{(n)v_1}\cdots\eta_r^{(n)v_r};\frac{1}{\mathfrak{a}\mathfrak{d}}\Big)-1\Big]u^{n(1-s)}\exp\{v^Tm\}dv_1\cdots dv_r\frac{du}{u}$$

with

$$b_m = \begin{cases}1/n & \text{if } m_1=\cdots=m_r=0,\\ 0 & \text{otherwise.}\end{cases}$$

If $m_1^2+\cdots+m_r^2>0$ the right-hand side of (6) is an integral function of $s$; if $m_1=\cdots=m_r=0$ there are two simple poles at $s=0$ and $s=1$. So we recognize that $\Phi_m(s,\mathfrak{a})$ is an integral function of $s$ except in the case $m_1=\cdots=m_r=0$; $\Phi_0(s,\mathfrak{a})$ has a simple pole at $s=1$. Another immediate consequence of equation (6) is the functional equation

$$\Phi_m(s,\mathfrak{a}) = \left(\frac{dN\mathfrak{a}^2}{\pi^n}\right)^{1-2s}\prod_{p=1}^{n}\frac{\Gamma(1-s+iE_p(m))}{\Gamma(s-iE_p(m))}\Phi_{-m}\left(1-s,\frac{1}{\mathfrak{a}\mathfrak{d}}\right), \tag{7}$$

which holds for all $m_1,\cdots,m_r$.

By equations (6) and (7) we can estimate the functions $\Phi_m(s,\mathfrak{a})$ uniformly in $m_1,\cdots,m_r$ in the infinite strip $-\epsilon\leqq\sigma\leqq1+\epsilon$, $\epsilon>0$. If we apply Phragmén-Lindelöf's extension of the maximum-modulus theorem to the functions $\Phi_m(s,\mathfrak{a})$ we obtain the inequalities:

$$\begin{aligned}|\Phi_m(\sigma+it,\mathfrak{a})| &\leqq c_4(\epsilon)\prod_{p=1}^{n}(1+|t-E_p(m)|)^{1-\sigma+\epsilon},\\ &-\epsilon\leqq\sigma\leqq1+\epsilon,\ m_1^2+\cdots+m_r^2>0.\end{aligned} \tag{8}$$

Inequality (8) also holds for $\Phi_0(s,\mathfrak{a})$ if $|t|\geqq c_5$. (The calculations which lead to (8) are given very explicitly for a similar case in [1].)

3. Now it is easy to investigate the asymptotic behaviour of the right-hand side of (4) for $(x_1\cdots x_n)\to\infty$. The path of integration in (4) is replaced by a straight line in the critical strip whose point of

intersection with the real axis may be $\sigma=\delta$, $0<\delta<1$. Considering the pole of $\Phi_0(s, \mathfrak{a})$ at $s=1$ we find:

$$
\begin{aligned}
J &= \frac{1}{2^n}\frac{\pi^n}{dN\mathfrak{a}^2}\prod_{p=1}^{n}\left[(y_p+x_p)^2-x_p^2\right] \\
(9)\qquad &\quad + \frac{n}{2\pi i\,|R|}\sum_{m_1,\cdots,m_r=-\infty}^{+\infty}\int_{\delta-i\infty}^{\delta+i\infty}\Phi_m(s,\mathfrak{a}) \\
&\qquad\qquad \cdot\prod_{p=1}^{n}\frac{(y_p+x_p)^{s+1-iE_p(m)}-x_p^{s+1-iE_p(m)}}{(s-iE_p(m))(s+1-iE_p(m))}\,ds, \\
&\qquad\qquad s=\delta+it,\ 0<\delta<1.
\end{aligned}
$$

The infinite sum in (9) can be easily estimated if one considers that the following determinant does not vanish for $1\leqq k\leqq n$:

$$
\begin{vmatrix}
e_k^{(1)}-e_1^{(1)} & \cdots & e_k^{(1)}-e_{k-1}^{(1)} & e_k^{(1)}-e_{k+1}^{(1)} & \cdots & e_k^{(1)}-e_n^{(1)} \\
\vdots & & \vdots & \vdots & & \vdots \\
e_k^{(r)}-e_1^{(r)} & \cdots & e_k^{(r)}-e_{k-1}^{(r)} & e_k^{(r)}-e_{k+1}^{(r)} & \cdots & e_k^{(r)}-e_n^{(r)}
\end{vmatrix}.
$$

Then we obtain from (9)

$$
(10)\quad J=\frac{1}{2^n}\frac{\pi^n}{dN\mathfrak{a}^2}\prod_{p=1}^{n}\left[(y_p+x_p)^2-x_p^2\right]+O\left(\prod_{p=1}^{n}(y_p+x_p)^{\delta+1}\right).
$$

If we choose

$$
y_p=x_p(x_1\cdots x_n)^{-1/(n+1)},\qquad p=1,\cdots,n
$$

and divide $J$ by the product $y_1\cdots y_n$ equation (10) yields for $x_1\cdots x_n\to\infty$ and any $\delta>0$

$$
(11)\quad \frac{J}{y_1\cdots y_n}=\left(\frac{\pi^n}{dN\mathfrak{a}^2}\right)(x_1\cdots x_n)+O((x_1\cdots x_n)^{n/(n+1)+\delta}).
$$

Recalling the remark in the beginning of §2 we observe that (11) also gives the asymptotic behaviour of $F(x_1,\cdots,x_n)$ for $x_1\cdots x_n\to\infty$ and any $\delta>0$:

$$
F(x_1,\cdots,x_n)=\left(\frac{\pi^n}{dN\mathfrak{a}^2}\right)(x_1\cdots x_n)+O((x_1\cdots x_n)^{n/(n+1)+\delta}).
$$

This proves the theorem formulated in the introduction.

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