THE COHOMOLOGY RING OF A COMPACT LIE GROUP WITH BI-INVARIANT METRIC

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I. **Introduction.** In this note we shall show that the adjoint operation * obtained from a bi-invariant riemannian metric on a compact Lie group induces an isomorphism between the cup and Pontrjagin products on the cohomology ring. This fact is easily and directly verifiable in the case of a torus, where, as we shall show elsewhere, it has interesting applications to the classical theory of abelian varieties; in fact, it motivates a definition of * on the numerical equivalence ring of an abstract polarized abelian variety.

II. **Algebraic preliminaries.** Let E be an n-dimensional vector space over $\mathbf{R}$, $\Lambda^p(E)$ the p-fold exterior product, and $\tilde{E}$, $\Lambda^p(E)^\sim$, their respective dual spaces. There is a canonical isomorphism i_p: $\Lambda^p(E)^\sim \to \Lambda^p(\tilde{E})$. An orientation of E is an isomorphism ϵ: $\Lambda^n(E) \cong \mathbf{R}$. It gives rise to a dual orientation $\tilde{\epsilon}$: $\Lambda^n(\tilde{E}) \cong \mathbf{R}$, and we denote the fundamental n-vector and n-covector by $e = \epsilon^{-1}(1)$ and $\tilde{e} = \tilde{\epsilon}^{-1}(1)$, respectively. One defines an isomorphism j_p: $\Lambda^p(E) \to \Lambda^{n-p}(E)^\sim$ by letting $j_p(\alpha)(\beta) = \epsilon(\alpha \wedge \beta)$. This gives an isomorphism $k_p = i_{n-p} \circ j_p$: $\Lambda^p(E) \cong \Lambda^{n-p}(\tilde{E})$. For $\alpha \in \Lambda^p(E)$, $\beta \in \Lambda^q(E)$, let $\alpha \vee \beta = k^{-1}(k\alpha \wedge k\beta) \in \Lambda^{p+q-n}(E)$, and for $\tilde{\beta} \in \Lambda^q(\tilde{E})$, $\alpha \barwedge \tilde{\beta} = \alpha \vee k^{-1}(\tilde{\beta}) \in \Lambda^{p-q}(E)$.

The composition map $T: E \oplus E \to E$, sending $\alpha \oplus \beta$ into $\alpha + \beta$, can be uniquely extended to an algebra homomorphism $\{T^p\}$, T^p: $\Lambda^p(E \oplus E) \to \Lambda^p(E)$. Since $\Lambda^p(E \oplus E) \cong \oplus_{r+s=p} \Lambda^r(E) \otimes \Lambda^s(E)$, we can define, for $\alpha \in \Lambda^r(E)$ and $\beta \in \Lambda^s(E)$, $T^{r+s}(\alpha \otimes \beta) \in \Lambda^{r+s}(E)$, and a simple computation shows this to equal $\alpha \wedge \beta$. Also, T: $E \oplus E \to E$ may be dualized to give $\tilde{T}$: $\tilde{E} \to \tilde{E} \oplus \tilde{E}$, which extends uniquely to an algebra homomorphism $\{\tilde{T}^p\}$, T^p: $\Lambda^p(\tilde{E}) \to \Lambda^p(\tilde{E} \oplus \tilde{E})$. Let $\tilde{e}' = T^n(\tilde{e}) \in \Lambda^n(\tilde{E} \oplus \tilde{E})$. Given $\alpha \in \Lambda^p(E)$, $\beta \in \Lambda^q(E)$, we have $\alpha \otimes \beta \in \Lambda^{p+q}(E \oplus E)$ and $\alpha \otimes \beta \barwedge \tilde{e}' \in \Lambda^{p+q-n}(E \oplus E)$. An easy computation shows that $T^{p+q-n}(\alpha \otimes \beta \barwedge \tilde{e}') = \alpha \vee \beta$.

A quadratic form on E is an isomorphism ϕ: $E \to \tilde{E}$, extendable uniquely to an algebra isomorphism $\{\phi^p\}$, ϕ^p: $\Lambda^p(E) \to \Lambda^p(\tilde{E})$. We define $*$: $\Lambda^p(E) \to \Lambda^{n-p}(E)$ by $*\alpha = k^{-1}\phi\alpha$. Note that $*(\alpha \wedge \beta) = k^{-1}(\phi\alpha \wedge \phi\beta) = k^{-1}(kk^{-1}\phi\alpha \wedge kk^{-1}\phi\beta) = k^{-1}(k*\alpha \wedge k*\beta) = *\alpha \vee *\beta$. So $*$ is an isomorphism of the $\wedge$-algebra onto the $\vee$-algebra; if ϕ is unitary, i.e., $*1 = e$, the map is unit preserving.

Received by the editors February 2, 1963.

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Needless to say, the above considerations extend to the case where E is a vector bundle over a manifold M; e.g., if E is the cotangent bundle of an oriented manifold, we recover the familiar * of Weitzenback-Hodge.

III. **Integration along the fibre.** Let G be a compact oriented Lie group with dual Lie algebra E (whose orientation is ϵ), and let $\Phi: Y^k \to X^m$ be a principal fibre bundle with group G, X and Y being compact and having orientations compatible with that of G (with respect to the local product structure). The orientation of E gives G a unique left-invariant Haar measure. For all $x \in X$, there is a homeomorphism $\psi_x: G \to \Phi^{-1}(x)$, sending the invariant n-vector $\tilde{e}$ onto an invariant n-vector $\tilde{e}_0$ along the fibre. Let α be a p-form on Y; we define a $(p-n)$-form ${}_*\Phi(\alpha)$ on X by integrating along the fibre. Precisely, $\alpha \overline{\wedge} \tilde{e}_0$, along the fibre $\Phi^{-1}(x)$ is a form annihilating all vectors tangent to the fibre; hence it may be integrated, by integrating its coefficients with respect to the Haar measure, to give a $(p-n)$-form ${}_*\Phi(\alpha)_x$ at x. The local product structure and the invariance of the measure guarantee that ${}_*\Phi(\alpha)$ is well defined and differentiable, and it is easy to check that ${}_*\Phi(\alpha)$ is closed if α is.

Let y be a p-cohomology class in Y, the de Rham class of a form α. Its Poincaré dual class $Py \in H_{k-p}(Y, \boldsymbol{R})$ is defined as a linear functional on $H^{k-p}(Y, \boldsymbol{R})$ by $Py(z) = \int_Y \alpha \wedge \beta$, z being the de Rham class of a form β. The map Φ induces a map $\Phi_*: H_{k-p}(Y, \boldsymbol{R}) \to H_{k-p}(X, \boldsymbol{R})$ by letting $\Phi_*(Py)(w) = \int_Y \alpha \wedge \Phi^* \gamma$, where w is the de Rham class of a $(k-p)$-form γ on X. On the other hand, the Poincaré dual class of ${}_*\Phi(\alpha)$ is defined as a functional on $H^{k-p}(X, \boldsymbol{R})$ by sending the de Rham class w of a form γ into $\int_X {}_*\Phi(\alpha) \wedge \gamma$. A partition of unity and Fubini's theorem show immediately that $P[{}_*\Phi(\alpha)] = \Phi_*(Py)$, y again being the class of α. Hence ${}_*\Phi$ induces on the de Rham cohomology the well-known Gysin homomorphism.

IV. **The convolution product.** Now let $X = G$, $Y = G \times G$ and Φ be the composition map. Further, let π_1 and π_2 be the projections of $G \times G$ onto its factors. If α is a p-form, and β a q-form, on G, the $(p+q-n)$-form $\alpha \bullet \beta = {}_*\Phi(\pi_1^* \alpha \wedge \pi_2^* \beta)$ will be called the convolution of α and β. (If $p = q = n$, $\alpha = fe$ and $\beta = ge$, then $\alpha \bullet \beta = he$, and it is clear that the function h is the convolution, in the usual sense, of the functions f and g.) From the last remarks in §III, it is clear that the convolution algebra on closed forms induces on the cohomology of G the algebra structure obtained by transposing (via Poincaré duality) the Pontrjagin algebra on homology to cohomology.

Now let α be right invariant, and β left invariant. Then $\pi_1^* \alpha$

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$\wedge\pi\beta_2^*\overline{\wedge}\tilde{e}_0$ is invariant along the fibre $\Phi^{-1}(1)$ so the value of $\alpha\bullet\beta$ at 1 is merely the image under the canonical map $\Lambda^{p+q-n}(E\oplus E)\to\Lambda^{p+q-n}(E)$ of the value of $\pi_1^*\alpha\wedge\pi_2^*\beta\tilde{e}\overline{\wedge}_0$ at 1×1. (We are using the fact that the map $\psi_1: G\to G\times G$ induces, on the dual Lie algebra, the canonical map of composition.) By the remarks in II, $\alpha\bullet\beta$ evaluated at 1 is thus (the value of α at 1) $\vee$ (the value of β at 1).

Now assume that both α and β are bi-invariant. Then $\pi_1^*\alpha\wedge\pi_2^*\beta$ is bi-invariant on $G\times G$; invariant, in particular, under the liftings via Φ of both right and left translations on G. Hence, $\alpha\bullet\beta$ is bi-invariant, and so equals $\alpha\vee\beta$.

V. **Conclusion of the proof.** We may map $H^*(G, \boldsymbol{R})$ into $\Lambda^*(E)$ by sending each class into the unique bi-invariant form it contains, evaluated at 1. It is well known that this is a homomorphism of the cup-algebra into the $\wedge$-algebra, and we have shown above that it is also a homomorphism of the (transposed) Pontrjagin algebra into the $\vee$-algebra.

Now, let ϕ be a bi-invariant riemannian metric on G. (One is easily constructed by integrating a positive-definite quadratic form on $\tilde{E}$ under the adjoint group, and then translating.) Then (e.g., since bi-invariant forms are harmonic) the action of * on $\Lambda^*(E)$ is compatible with the action of * on cohomology. Letting $\oplus$ denote the (cohomological) Pontrjagin product, we may combine this last remark with that at the end of §II to assert that, for any cohomology classes x and y on G, $*(x\cup y)=(*x)\oplus(*y)$.

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