AN ERGODIC LEMMA

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Recently, A. Brunel proved an ergodic lemma [1] and applied it to give another proof of the following theorem [2].

THEOREM 1. *Let* $(X, \mathfrak{F}, \mu)$ *be a* σ*-finite measure space and let* T *be a positive linear operator in* $L_1(X, \mathfrak{F}, \mu)$ *with* $\|T\| \leqq 1$. *Then for any* $f \in L_1$, $p \in L_1$, $p \geqq 0$,

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} T^k f}{\sum_{k=0}^{n} T^k p} \tag{1}$$

exists and is finite a.e. on $\{x \mid 0 < \sum_{k=0}^{\infty} T^k p(x) \leqq \infty\}$.

In the present note we show that a result which is stronger than Brunel's lemma can be obtained from Lemmas 1 and 2 of [2], which are the essential two lemmas used in [2] to prove Theorem 1. These two lemmas are also needed to identify the limit (1), as is shown in [3]. It would seem, therefore, that in order to give a complete set of results by the shortest and most direct route one should start with Lemmas 1 and 2 of [2], obtain the existence of the limit (1) by Lemma 3 of the present note and then identify this limit following the method of [3].

We state Lemmas 1 and 2 of [2], in a slightly modified form, as follows.

LEMMA 1. *Let* T *be an operator satisfying the hypotheses of Theorem* 1. *Let* $F = g - h$, $g \geqq 0$, $h \geqq 0$, $g \in L_1$, $h \in L_1$ *and*

$$\sup_{n \geqq 0} \sum_{k=0}^{n} T^k F > 0 \quad \textit{a.e. on a set } B.$$

Then there exist sequences, $\{d_k\}$ *and* $\{g_k\}$, *of non-negative functions in* L_1, *such that*

(i) $\sum_{k=0}^{n} \int d_k + \int g_n \leqq \int g$,

(ii) $\sum_{k=0}^{\infty} d_k \leqq h$ *a.e. and* $\sum_{k=0}^{\infty} d_k = h$ *a.e. on* B,

(iii) $T^n g = \sum_{k=0}^{n} T^{n-k} d_k + g_n$.

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LEMMA 2. *Let T be an operator satisfying the hypotheses of Theorem* 1. *Then for any $g\in L_1$, $h\in L_1$, $h\geqq 0$ and for any integer j we have*

$$\lim_{n\to\infty} \frac{T^{n+j}g}{\sum_{k=0}^{n} T^k h} = 0, \qquad n \geqq \max(0, -j),$$

a.e. on $\left\{x \,\middle|\, 0<\sum_{k=0}^{\infty} T^k h(x) \leqq \infty\right\}$.

From these two lemmas we obtain the following:

LEMMA 3. *Let T satisfy the hypotheses of Theorem* 1. *For any $f\in L_1$, define*

$$E_f' = \left\{x \,\middle|\, \limsup_{n\to\infty} \sum_{k=0}^{n} T^k f(x) > 0\right\}.$$

Then, to any measurable set E, one can associate a function $\psi_E\in L_\infty$ such that

$$E \subset E_f' \quad \textit{implies} \quad \int \psi_E f \geqq 0.$$

Brunel's Lemma is obtained from this lemma if the set E_f' is replaced by the set E_f defined as

$$E_f = \left\{x \,\middle|\, \sup_{n\geqq m} \sum_{k=m}^{n} T^k f(x) > 0, \text{ for all } m \geqq 0\right\}.$$

It is clear that $E_f\subset E_f'$.

PROOF OF LEMMA 3. Define ψ_E as follows [1]:

$$\psi_E(x) = \lim_{n\to\infty} \psi_E^{(n)}(x),$$

with $\psi_E^{(0)}=\chi$, $\psi_E^{(n)}=\chi\vee S\psi^{(n-1)}$, $n\geqq 1$, where χ is the characteristic function of E, and S: $L_\infty\to L_\infty$ denotes the adjoint transformation of T. Since $S\geqq 0$ and $\|S\|_\infty\leqq 1$, we obtain that, for any $f\in L_1$,

$$\int \psi_E f = \int \sum_{k=0}^{\infty} \alpha_k = \sum_{k=0}^{\infty} \int \alpha_k,$$

where the sequence $\{\alpha_k\}$ is defined as

$$(2) \qquad \begin{aligned} \alpha_0 &= \chi f; & \beta_0 &= \chi' f, \\ \alpha_k &= \chi T\beta_{k-1}; & \beta_k &= \chi' T\beta_{k-1}, \quad k \geqq 1, \end{aligned}$$

$\chi' = 1 - \chi$ being the characteristic function of the complement of E. With usual notations, let $f = f^+ - f^-$ and let $\{\alpha_i^{(+)}\}$, $\{\beta_i^{(+)}\}$, $\{\alpha_i^{(-)}\}$, $\{\beta_i^{(-)}\}$ be the sequences which are defined as above, but starting from f^+ and f^-, respectively. To prove the lemma it is sufficient to show that

$$\sum_{i=0}^{j} \int \alpha_i^{(-)} \leqq a \sum_{i=0}^{\infty} \int \alpha_i^{(+)}, \tag{3}$$

for any integer $j \geqq 0$, and for any real number $a > 1$.

Let $g = af^+$ and $h = \sum_{i=0}^{j} \alpha_i^{(-)}$, where $a > 1$ and $j \geqq 0$ are fixed, and consider $F = g - h$. We will show that

$$\sup_{n \geqq 0} \sum_{k=0}^{n} T^k F > 0 \quad \text{a.e. on } E. \tag{4}$$

First note that

$$\chi T^n f^- = \chi \sum_{k=0}^{n} T^{n-k} \alpha_k^{(-)},$$

which follows from definition (2). Therefore, using the positiveness of T and f^-, we have

$$\chi \sum_{k=0}^{n} T^k h = \chi \sum_{k=0}^{n} T^k \sum_{i=0}^{j} \alpha_i^{(-)} \leqq \chi \sum_{k=0}^{n+j} T^k f^-. \tag{5}$$

Now (4) is trivial on the set where $\sum_{k=0}^{\infty} T^k h = 0$. On the set $E \cap \{x \mid 0 < \sum_{k=0}^{\infty} T^k h\}$, by virtue of (5),

$$\limsup_{n\to\infty} \frac{\sum_{k=0}^{n} T^k g}{\sum_{k=0}^{n} T^k h} \geqq \limsup_{n\to\infty} \frac{\sum_{k=0}^{n} T^k g}{\sum_{k=0}^{n+j} T^k f^-} = a \limsup_{n\to\infty} \frac{\sum_{k=0}^{n+j} T^k f^+}{\sum_{k=0}^{n+j} T^k f^-},$$

where the last equality follows from Lemma 2. But

$$\limsup_{n\to\infty} \left[\frac{\sum_{k=0}^{n} T^k f^+}{\sum_{k=0}^{n} T^k f^-} - 1 \right] = \limsup_{n\to\infty} \frac{\sum_{k=0}^{n} T^k f}{\sum_{k=0}^{n} T^k f^-} \geqq 0,$$

from the hypothesis that $E \subset E_f'$. Hence,

$$\limsup_{n\to\infty} \frac{\sum_{k=0}^{n} T^k g}{\sum_{k=0}^{n} T^k h} > 1 \quad \text{or} \quad \limsup_{n\to\infty} \frac{\sum_{k=0}^{n} T^k F}{\sum_{k=0}^{n} T^k h} > 0$$

which implies (4). Therefore we can apply Lemma 1 to F. Since $\chi h = h$, $\chi' h = 0$, the d_k's are zero outside E and assertion (ii) of Lemma 1 yields

$$\sum_{k=0}^{\infty} \int d_k = \int h = \sum_{i=0}^{j} \int \alpha_i^{(-)}. \tag{6}$$

The proof will be concluded by showing that

$$\sum_{k=0}^{n} \int d_k \leqq a \sum_{k=0}^{n} \int \alpha_k^{(+)}, \quad \text{for all } n \geqq 0, \tag{7}$$

i.e., that (6) implies (3).

To prove (7) first note that

$$\begin{aligned} g &= d_0 + g_0, \\ Tg_n &= d_{n+1} + g_{n+1}, \qquad n \geqq 0, \end{aligned}$$

which follows from (iii). Similarly definitions of $\alpha_n^{(+)}$, $\beta_n^{(+)}$ and g imply that

$$\begin{aligned} g &= a\alpha_0^{(+)} + a\beta_0^{(+)}, \\ aT\beta_n^{(+)} &= a\alpha_{n+1}^{(+)} + a\beta_{n+1}^{(+)}. \end{aligned}$$

Hence

$$\begin{aligned} g_0 - a\beta_0^{(+)} &= a\alpha_0^{(+)} - d_0, \\ g_{n+1} - a\beta_{n+1}^{(+)} &= T(g_n - a\beta_n^{(+)}) + (a\alpha_{n+1}^{(+)} - d_{n+1}), \end{aligned}$$

and

$$\begin{aligned} \chi'(g_0 - a\beta_0^{(+)}) &= 0, \\ \chi'(g_{n+1} - a\beta_{n+1}^{(+)}) &= \chi' T(g_n - a\beta_n^{(+)}); \end{aligned}$$

also

$$\chi(g_n - a\beta_n^{(+)}) = \chi g_n \geqq 0, \qquad n \geqq 0,$$

which shows that

$$g_n - a\beta_n^{(+)} \geqq 0, \qquad n \geqq 0. \tag{8}$$

Now

$$0 = \int [(a\alpha_0^{(+)} - d_0) - (g_0 - a\beta_0^{(+)})],$$

and

$$\int \left[\sum_{k=0}^{n} (a\alpha_k^{(+)} - d_k) - (g_n - a\beta_n^{(+)}) \right]$$

$$\leqq \int \left[\sum_{k=0}^{n} (a\alpha_k^{(+)} - d_k) - T(g_n - a\beta_n^{(+)}) \right]$$

$$= \int \left[\sum_{k=0}^{n+1} (a\alpha_k^{(+)} - d_k) - (g_{n+1} - a\beta_{n+1}^{(+)}) \right],$$

or, by induction,

$$0 \leqq \int \left[\sum_{k=0}^{n} (a\alpha_k^{(+)} - d_k) - (g_n - a\beta_n^{(+)}) \right],$$

for all $n \geqq 0$, which proves (7).

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