A GENERALIZED EIGENFUNCTION EXPANSION OF THE GREEN'S FUNCTION

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We consider the problem

$$(1) \qquad (L - \lambda)x = 0, \qquad Ux = 0,$$

where L is the ordinary differential operator given by

$$Lx(t) = p_0(t)x^{(n)} + p_1(t)x^{(n-1)} + \cdots + p_n(t)x, \qquad p_0(t) \neq 0 \text{ on } [a, b],$$

and U is the vector boundary form given by

$$U_i x(t) = \sum_{j=1}^{n} (a_{ij}x^{(j-1)}(a) + b_{ij}x^{(j-1)}(b)), \qquad i = 1, \cdots, n.$$

Here the $p_j(t)$ are complex-valued functions in $C^{n-j}[a, b]$, and the a_{ij} and b_{ij} are complex. It is no restriction to take $p_0(t) \equiv 1$. Corresponding to (1) we have the adjoint problem

$$(L^+ - \bar{\lambda})y = 0, \qquad U^+ y = 0,$$

where L^+ and U^+ are determined by the usual requirement that, for all $x(t)$, $y(t) \in C^n[a, b]$ such that $Ux = 0$, $U^+y = 0$,

$$\int_a^b \bar{y} Lx \, dt = \int_a^b x[L^+ y]^- \, dt.$$

Let $G(t, \tau, \lambda)$ be the Green's function for (1). Then, if (1) is selfadjoint, we have the well-known expansion

$$(2) \qquad G(t, \tau, \lambda) = \sum_{n=1}^{\infty} \frac{x_n(t)\bar{x}_n(\tau)}{\lambda_n - \lambda},$$

where λ_n and x_n are the eigenvalues and eigenfunctions of (1), respectively. In this paper we generalize the expansion to the nonselfadjoint case. In our expansion we will make use of generalized eigenfunctions. A generalized eigenfunction of rank r for (1) is any solution of

$$(L-\lambda)^r x = 0, \ (L-\lambda)^{r-1}x \neq 0, \ Ux = 0, \ U(L-\lambda)x = 0, \ \cdots, \ U(L-\lambda)^{r-1}x = 0.$$

If x is of rank r, then x, $(L-\lambda)x, \cdots, (L-\lambda)^{r-1}x$ form a chain of

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generalized eigenfunctions of decreasing rank. Thus if $\lambda_1, \lambda_2, \cdots$ are the eigenvalues of (1) ordered, say, by increasing modulus, the space of generalized eigenfunctions for λ_j of rank less than or equal to m_j is the solution space of

(3) $(L - \lambda_j)^{m_j}x = 0, Ux = 0, U(L - \lambda_j)x = 0, \cdots, U(L - \lambda_j)^{m_j-1}x = 0.$

There exists a basis of (3) made up of chains of generalized eigenfunctions, and all such chain bases have the same number and lengths of chains [1]. We denote such a chain basis by

(4) $\overset{(j)}{x_{kl}}(t), \quad j = 1, 2, \cdots; k = 1, 2, \cdots, c_j; l = 1, 2, \cdots, r_{jk}.$

Here the first subscript denotes chain number, the second denotes rank, c_j is the number of chains corresponding to λ_j, r_{jk} is the length of the kth chain for λ_j. We assume the chains so ordered that $m_j \geqq r_{j1} \geqq r_{j2} \geqq \cdots \geqq r_{jc_j}$. Thus $(L-\lambda_j)x_{kl}^{(j)} = x_{k,l-1}^{(j)}$, $j=1, 2, \cdots; k=1, 2, \cdots, c_j; l=1, 2, \cdots, r_{jk}$. (We take $x_{k0}^{(j)} \equiv 0$).

Corresponding to (3) we have the adjoint problem

(5) $(L^+ - \bar{\lambda}_j)^{m_j}\psi = 0, U^+\psi = 0, U^+(L^+ - \bar{\lambda}_j)\psi = 0, \cdots, U^+(L^+ - \bar{\lambda}_j)^{m_j-1}\psi = 0,$

with the adjoint basis $\psi_{kl}^{(j)}(t)$. The ψ basis agrees with the x basis in rank, number, and lengths of chains. In particular, there exists the corresponding chain basis to (4) such that

(6) $$\int_a^b \overset{(j)}{x_{mn}}(t)\overset{(j)}{\bar{\psi}_{m'n'}}(t)\,dt = \delta_{mm'}\delta_{n,r_{jm}+1-n'} \text{ (i.e., the scalar product).}$$

(Such a corresponding chain basis may be obtained by starting with any ψ basis and using (6) to obtain the appropriate linear combinations.)

Our generalized expansion (indeed, the very fact that an infinite sequence of eigenvalues exist) will hold for what G. D. Birkhoff [2] has called regular boundary conditions. To define regularity we first normalize the boundary form U as follows:

Reduce the number of conditions $U_ix=0$ of order $n-1$ (that is, containing either $x^{(n-1)}(a)$ or $x^{(n-1)}(b)$) to a minimum, at most two, by linear combination. Then, in the remaining conditions, reduce those of order $n-2$ to a minimum, at most two, again by linear combination. Continue in this way as long as conditions remain. The normalized conditions will have the form

$$U_ix = U_{ia}x + U_{ib}x = 0, \qquad i = 1, \cdots, n,$$

where

$$U_{ia}x = \alpha_i x^{(k_i)}(a) + \sum_{j=0}^{k_i-1} \alpha_{ij} x^{(j)}(a), \quad U_{ib}x = \beta_i x^{(k_i)}(b) + \sum_{j=0}^{k_i-1} \beta_{ij} x^{(j)}(b),$$

$$n - 1 \geqq k_1 \geqq k_2 \geqq \cdots \geqq k_n,$$

where no three consecutive k's are the same.

DEFINITION. Let ω_i, $i=1, \cdots, n$, be the n roots of unity. Let the ω_i be taken in every order such that, for some $\rho \neq 0$,

$$\operatorname{Re}(\rho\omega_1) < \operatorname{Re}(\rho\omega_2) < \cdots < \operatorname{Re}(\rho\omega_n) \quad (\operatorname{Re} x \text{ denotes the real part of } x).$$

Then, if $n=2\mu-1$ is odd and always neither $\theta_0=0$ nor $\theta_1=0$, where θ_0 and θ_1 are defined by the identity

$$\theta_0 + \theta_1 s \equiv \begin{vmatrix} \alpha_1\omega_1^{k_1} & \cdots & \alpha_1\omega_{\mu-1}^{k_1} & (\alpha_1+s\beta_1)\omega_\mu^{k_1} & \beta_1\omega_{\mu+1}^{k_1} & \cdots & \beta_1\omega_n^{k_1} \\ \alpha_2\omega_1^{k_2} & \cdots & \alpha_2\omega_{\mu-1}^{k_2} & (\alpha_2+s\beta_2)\omega_\mu^{k_2} & \beta_2\omega_{\mu+1}^{k_2} & \cdots & \beta_2\omega_n^{k_2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_n\omega_1^{k_n} & \cdots & \alpha_n\omega_{\mu-1}^{k_n} & (\alpha_n+s\beta_n)\omega_\mu^{k_n} & \beta_n\omega_{\mu+1}^{k_n} & \cdots & \beta_n\omega_n^{k_n} \end{vmatrix},$$

or if $n=2\mu$ is even and always neither $\theta_1=0$ nor $\theta_2=0$, where θ_0, θ_1, and θ_2 are defined by the identity

$$\theta_0 + \theta_1 s + \frac{\theta_2}{s}$$

$$\equiv \begin{vmatrix} \alpha_1\omega_1^{k_1} & \cdots & \alpha_1\omega_{\mu-1}^{k_1} & (\alpha_1+s\beta_1)\omega_\mu^{k_1} & \left(\alpha_1+\dfrac{1}{s}\beta_1\right)\omega_{\mu+1}^{k_1} & \beta_1\omega_{\mu+2}^{k_1} & \cdots & \beta_1\omega_n^{k_1} \\ \alpha_2\omega_1^{k_2} & \cdots & \alpha_2\omega_{\mu-1}^{k_2} & (\alpha_2+s\beta_2)\omega_\mu^{k_2} & \left(\alpha_2+\dfrac{1}{s}\beta_2\right)\omega_{\mu+1}^{k_2} & \beta_2\omega_{\mu+2}^{k_2} & \cdots & \beta_2\omega_n^{k_2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_n\omega_1^{k_n} & \cdots & \alpha_n\omega_{\mu-1}^{k_n} & (\alpha_n+s\beta_n)\omega_\mu^{k_n} & \left(\alpha_n+\dfrac{1}{s}\beta_n\right)\omega_{\mu+1}^{k_n} & \beta_n\omega_{\mu+2}^{k_n} & \cdots & \beta_n\omega_n^{k_n} \end{vmatrix},$$

the conditions $Ux=0$ are said to be regular. For $n=2$ the standard Sturm-Liouville conditions $hx(a)+kx'(a)=0, lx(b)+mx'(b)=0$, where $mk\neq 0$, are regular. The conditions $x(a)=0$, $x'(a)=kx(b)$ are non-regular.

Our result is summarized in the following theorem.

THEOREM. *Let* (1) *be regular with the Green's function* $G(t, \tau, \lambda)$. *Let* $\lambda_1, \lambda_2, \cdots$ *be the eigenvalues of* (1) *ordered by increasing modulus, with* m_j *the order of* λ_j *as a pole of* G. *Let* $x_{kl}^{(j)}(t)$ *be a chain basis of* (3). *Let* $\psi_{kl}^{(j)}(t)$ *be the corresponding chain basis of* (5). *Then*

$$G(t,\tau,\lambda)=\sum_{j=1}^{\infty}\sum_{k=1}^{c_j}\sum_{l=1}^{r_{jk}}x_{kl}^{(j)}(t)\left[\frac{\bar\psi_{kl'}^{(j)}(\tau)}{\lambda_j-\lambda}-\frac{\bar\psi_{k,l'-1}^{(j)}(\tau)}{(\lambda_j-\lambda)^2}+-\cdots+(-1)^{l'-1}\frac{\bar\psi_{k1}^{(j)}(\tau)}{(\lambda_j-\lambda)^{l'}}\right],\quad t,\tau\in(a,b), \tag{7}$$

where $l'=r_{jk}+1-l$.

REMARK. When (1) is selfadjoint, all the $m_j=1$, $\psi_{k1}^{(j)}=x_{k1}^{(j)}$, and the expansion reduces to (2).

PROOF. We make use of two theorems. The first is due to G. D. Birkhoff.

THEOREM (BIRKHOFF [2]). *Let* (1) *be regular. If* $f(t)$ *is piecewise continuous with piecewise continuous derivative on* $[a,b]$, *then*

$$\tfrac{1}{2}[f(t+0)+f(t-0)]=\lim_{n\to\infty}-\frac{1}{2\pi i}\int_a^b dt'f(t')\oint_{C_n}G(t,t',\lambda)\,d\lambda \quad on\ (a,b). \tag{8}$$

At $t=a$ *and* $t=b$ *the sequence converges to linear combinations of* $f(a+0)$ *and* $f(b-0)$, *the coefficients being independent of* f. *Here* C_n *is a circle about the origin of radius* n *in the* λ-*plane.*

The second theorem concerns the residue of G.

THEOREM [1]. *Let* $G(t,t',\lambda)$ *be the Green's function of* (1). *If* λ_j *is a pole of* G *of order* m_j, *and* $x_{kl}^{(j)}(t)$ *constitute any chain basis of* (3), *then the residue of* G *at* λ_j *is*

$$-\sum_{k=1}^{c_j}\sum_{l=1}^{r_{jk}}x_{kl}^{(j)}(t)\bar\psi_{kl'}^{(j)}(t'),\qquad l'=r_{jk}+1-l, \tag{9}$$

where $\psi_{kl}^{(j)}(t)$ *is the corresponding chain basis of* (5).

Taking $f(t)$ in (8) equal to $G(t,\tau,\lambda)$ and using (9) we have

$$\begin{aligned}G(t,\tau,\lambda)&=\lim_{n\to\infty}-\frac{1}{2\pi i}\int_a^b dt'G(t',\tau,\lambda)\oint_{C_n}G(t,t',\lambda)\,d\lambda\\&=\sum_{j=1}^{\infty}\sum_{k=1}^{c_j}\sum_{l=1}^{r_{jk}}x_{kl}^{(j)}(t)\int_a^b G(t',\tau,\lambda)\bar\psi_{kl'}^{(j)}(t')\,dt',\quad t,\tau\in(a,b).\end{aligned} \tag{10}$$

We now evaluate the integral in (10). Since $G(t',\tau,\lambda)=\bar G^{+}(\tau,t',\bar\lambda)$, where G^{+} is the Green's function of the adjoint to problem (1) [3],

$$\int_a^b G(t',\tau,\lambda)\bar\psi_{kl'}^{(j)}(t')\,dt'=\left[\int_a^b G^{+}(\tau,t',\bar\lambda)\psi_{kl'}^{(j)}(t')\,dt'\right]^{-}. \tag{11}$$

If we define $\psi_{k0}^{(j)} \equiv 0$, and if λ is not an eigenvalue of (1),

$$(L^+ - \bar\lambda)\psi_{kl'}^{(j)}(t) = (L^+ - \bar\lambda_j)\psi_{kl'}^{(j)}(t) + (\bar\lambda_j - \bar\lambda)\psi_{kl'}^{(j)}(t)$$
$$= \psi_{k,l'-1}^{(j)}(t) + (\bar\lambda_j - \bar\lambda)\psi_{kl'}^{(j)}(t).$$

Thus

$$\psi_{kl'}^{(j)}(t) = \int_a^b G^+(t, t', \bar\lambda)\psi_{k,l'-1}^{(j)}(t')\, dt' + (\bar\lambda_j - \bar\lambda)\int_a^b G^+(t, t', \bar\lambda)\psi_{kl'}^{(j)}(t')\, dt'.$$

Thus

$$\int_a^b G^+(t, t', \bar\lambda)\psi_{kl'}^{(j)}(t')\, dt'$$
$$= \frac{\psi_{kl'}^{(j)}(t)}{\bar\lambda_j - \bar\lambda} - \frac{1}{\bar\lambda_j - \bar\lambda}\int_a^b G^+(t, t', \bar\lambda)\psi_{k,l'-1}^{(j)}(t')\, dt'$$
$$(12) \qquad = \frac{\psi_{kl'}^{(j)}(t)}{\bar\lambda_j - \bar\lambda} - \frac{\psi_{k,l'-1}^{(j)}(t)}{(\bar\lambda_j - \bar\lambda)^2} + \frac{1}{(\bar\lambda_j - \bar\lambda)^2}\int_a^b G^+(t, t', \bar\lambda)\psi_{k,l'-2}^{(j)}(t')\, dt'$$
$$= \frac{\psi_{kl'}^{(j)}(t)}{\bar\lambda_j - \bar\lambda} - \frac{\psi_{k,l'-1}^{(j)}(t)}{(\bar\lambda_j - \bar\lambda)^2} + - \cdots + (-1)^{l'-1}\frac{\psi_{k1}^{(j)}(t)}{(\bar\lambda_j - \bar\lambda)^{l'}}.$$

From (10), (11), and (12), (7) follows immediately.

This result is clearly anticipated in B. Friedman's *Principles and techniques of applied mathematics.*

References

1. M. Machover, *Generalized eigenvectors and separation of variables*, Doctoral Dissertation, New York University, New York, 1963.

2. G. D. Birkhoff, *Boundary value and expansion problems of ordinary linear differential equations*, Trans. Amer. Math. Soc. 9 (1908), 373–395; also in *Collected Mathematical Papers of G. D. Birkhoff*, Vol. I, Amer. Math. Soc., Providence, R.I., 1950.

3. E. A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, 1955; Chapters 7, 11, and 12.

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