PROXIMAL RELATIONS IN TOPOLOGICAL DYNAMICS

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In this note we shall prove that when the proximal relation of a transformation group \((X, T, \pi)\) with compact phase space \(X\) is transitive (i.e., it is an equivalence relation), then it is equivalent with the syndetically proximal relation. This would answer two questions in [4, Remark 5].

Standing notations. Let \((X, T, \pi)\) be a transformation group with compact phase space. The proximal relation of \((X, T, \pi)\) is denoted by \(P(X)\) and the syndetically proximal relation by \(L(X)\). The product transformation group induced by \((X, T, \pi)\) will be denoted by \((X \times X, T, \rho)\), which is defined by \((x, y)\rho^t = (x\pi^t, y\pi^t)\) for \((x, y) \in X \times X\) and \(t \in T\). For simplicity we shall write \(xt\) for \(x\pi^t\) and \((xt, yt) = (x, y)t\) for \((x, y)\rho^t\).

Reference. The proximal relation was studied in [1], [2], [3], [4]. The syndetically proximal relation was defined and studied in [4].

Proposition. If \(P(X)\) is transitive, then \(P(X) = L(X)\).

Proof. Since \(P(X)\) is transitive, so is \(P(X \times X)\) [1]. Then each orbit closure \(Cl(x, y)T\) in \((X \times X, T, \rho)\) contains a unique minimal set. Let \((x, y) \in P(X)\). If \(Cl(x, y)T \neq P(X)\), then there is \((a, b) \in Cl(x, y)T\). Let \(M\) be the (unique) minimal set contained in \(Cl(a, b)T\). There are two cases.

Case 1. \(M \cap P(X) = \emptyset\). By Lemma 2 of [1], there is a point \((u, v) \in M\) such that \((x, y), (u, v)) \in P(X \times X). This shows that \((x, u) \in P(X), (y, v) \in P(X)\), a fortiori, \((u, v) \in P(X)\) by the transitivity of \(P(X)\). We have the contradiction.

Case 2. \(M \cap P(X) \neq \emptyset\). By the definition of \(P(X)\), if \((x', y') \in P(X)\) and \(N\) is the minimal set contained in \(Cl(x', y')T\), then \(N \subset \Delta(X)\), the diagonal of \(X \times X\). This shows that \(M \subset \Delta(X)\), a fortiori, \((a, b) \in P(X)\). We have the contradiction also. Hence, \(Cl(x, y)T \subset P(X)\) when \((x, y) \in P(X)\). By Lemma 5 of [4], \(P(X) \subset L(X)\), \(P(X) = L(X)\).

Corollary. \(P(X)\) is an equivalence relation if and only if \(P(X \times X) = P(X) \times P(X)\).

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Let $X$ be a set and $U$ a uniformity on $X$. We will show that if $(X, U)$ is totally bounded, every net in $X$ has a Cauchy subnet. For each $d \in U$, let $S^d_1, \ldots, S^d_k$ be a finite covering of $X$ by $d$-spheres. Let $T_d$ be the topology on $X$ having $S^d_1, \ldots, S^d_k$ as its subbasis. Clearly the space $(X, T_d)$ is compact. Therefore, $Y = \prod_{d \in U} (X, T_d)$ is compact.

Now, let $(p_i)$ be a net in $X$. Then $\Delta \circ (p_i)$ is a net in $Y$, where $\Delta : X \to Y$ is the diagonal. By compactness, there exists a convergent subnet, $(q_j)$, of $\Delta \circ (p_i)$. Then $\Delta^{-1} \circ (q_j)$ is a subnet of $(p_i)$ which is clearly Cauchy.

Thus, if $(X, U)$ is also complete, every net in $X$ has a convergent subnet, so $(X, U)$ is compact.

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