PROXIMAL RELATIONS IN TOPOLOGICAL DYNAMICS[1]

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In this note we shall prove that when the proximal relation of a transformation group (X, T, π) with compact phase space X is transitive (i.e., it is an equivalence relation), then it is equivalent with the syndetically proximal relation. This would answer two questions in [4, Remark 5].

Standing notations. Let (X, T, π) be a transformation group with compact phase space. The proximal relation of (X, T, π) is denoted by $P(X)$ and the syndetically proximal relation by $L(X)$. The product transformation group induced by (X, T, π) will be denoted by $(X\times X, T, \rho)$, which is defined by $(x, y)\rho^t=(x\pi^t, y\pi^t)$ for $(x, y) \in X\times X$ and $t\in T$. For simplicity we shall write xt for $x\pi^t$ and $(xt, yt) =(x, y)t$ for $(x, y)\rho^t$.

Reference. The proximal relation was studied in [1], [2], [3], [4]. The syndetically proximal relation was defined and studied in [4].

PROPOSITION. *If $P(X)$ is transitive, then $P(X)=L(X)$.*

PROOF. Since $P(X)$ is transitive, so is $P(X\times X)$ [1]. Then each orbit closure $\mathrm{Cl}(x, y)T$ in $(X\times X, T, \rho)$ contains a unique minimal set. Let $(x, y)\in P(X)$. If $\mathrm{Cl}(x, y)T-P(X)\neq\square$, then there is $(a, b) \in\mathrm{Cl}(x, y)T-P(X)$. Let M be the (unique) minimal set contained in $\mathrm{Cl}(a, b)T$. There are two cases.

Case 1. $M\cap P(X)=\square$. By Lemma 2 of [1], there is a point $(u,v)\in M$ such that $((x, y), (u, v))\in P(X\times X)$. This shows that $(x, u)\in P(X)$, $(y, v)\in P(X)$, a fortiori, $(u, v)\in P(X)$ by the transitivity of $P(X)$. We have the contradiction.

Case 2. $M\cap P(X)\neq\square$. By the definition of $P(X)$, if $(x', y')\in P(X)$ and N is the minimal set contained in $\mathrm{Cl}(x', y')T$, then $N\subset\Delta(X)$, the diagonal of $X\times X$. This shows that $M\subset\Delta(X)$, a fortiori, $(a, b) \in P(X)$. We have the contradiction also. Hence, $\mathrm{Cl}(x, y)T\subset P(X)$ when $(x, y)\in P(X)$. By Lemma 5 of [4], $P(X)\subset L(X)$, $P(X)=L(X)$.

COROLLARY. *$P(X)$ is an equivalence relation if and only if $P(X\times X) =P(X)\times P(X)$.*

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A TOTALLY BOUNDED, COMPLETE UNIFORM SPACE IS COMPACT

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Let X be a set and U a uniformity on X. We will show that if (X, U) is totally bounded, every net in X has a Cauchy subnet. For each $d \in U$, let $S_d^1, \cdots, S_d^n$ be a finite covering of X by d-spheres. Let T_d be the topology on X having $S_d^1, \cdots, S_d^n$ as its subbasis. Clearly the space (X, T_d) is compact. Therefore, $Y = \prod_{d \in U} (X, T_d)$ is compact.

Now, let (p_i) be a net in X. Then $\Delta \circ (p_i)$ is a net in Y, where $\Delta: X \to Y$ is the diagonal. By compactness, there exists a convergent subnet, (q_j), of $\Delta \circ (p_i)$. Then $\Delta^{-1} \circ (q_j)$ is a subnet of (p_i) which is clearly Cauchy.

Thus, if (X, U) is also complete, every net in X has a convergent subnet, so (X, U) is compact.

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