ON QUASI-CONVERGENCE OF SERIES OF INDEPENDENT RANDOM VARIABLES[1]

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1. **Statement of the result.** The purpose of this paper is to prove the following theorem.

THEOREM. *Let* $\{X_n\}$ *be a sequence of independent random variables. A necessary and sufficient condition that there exists a sequence of real numbers* $\{\lambda_n\}$ *such that* $\sum_{n=1}^{\infty}(X_n-\lambda_n)$ *converges a.s. is that there exist real numbers* $\{a_{ij},\ i, j=1, 2, \cdots\}$ *such that*

(i) $a_{ij}\to 1$ *as* $i\to\infty$, *for every* j,

(ii) $S_i=\sum_{j=1}^{\infty} a_{ij}X_j$ *converges a.s., for every* i, *and*

(iii) $S_i\to$*(some random variable)* S *a.s. as* $i\to\infty$.

The sufficient part of this theorem was proved by J. Marcinkiewicz and A. Zygmund [**3**]. An entirely different proof of this is given here by means of concentration functions and, in particular, by using a theorem due to K. Ito ([**1**, p. 46], and restated and proved in a different manner in [**4**]). The statement and proof that the condition given in the theorem is necessary appears to be new. In §2 a lemma is proved which is used to prove that the condition is sufficient; besides its use in this paper this lemma should find considerable application. In §3 the theorem is proved. It should be pointed out that no use is made of the independence of the random variables in the proof that the condition is necessary.

2. **A lemma.** A particular case (in a sense) of the lemma given in this section was stated but not proved by P. Lévy [**2**, p. 134] and was stated and proved in [**4**] (see Lemma 4, pp. 719, 720). Since this particular case is not general enough to use in the proof that the condition given in the theorem is sufficient, a more general lemma must be stated and proved.

Let us recall that if F is a probability distribution function, then its concentration function Q_F is defined by

$$Q_F(L) = \sup_x\{F(x+L+0) - F(x-0)\},$$

for all $L\geqq 0$. For some of the known properties of concentration functions the reader is referred to [**2**] and [**4**].

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LEMMA. *Let F, F_1, F_2, $\cdots$ be a sequence of distribution functions, and let Q, Q_1, Q_2, $\cdots$ be the corresponding concentration functions. If $F_n(x)\to F(x)$ as $n\to\infty$ at all x at which F is continuous, then $Q_n(x)\to Q(x)$ as $n\to\infty$ at all x at which Q is continuous.*

PROOF. Let L be a point at which Q is continuous. It suffices to prove that

$$\limsup_{n\to\infty} Q_n(L) \leqq Q(L) \quad \text{if } L \geqq 0 \tag{1}$$

and

$$\liminf_{n\to\infty} Q_n(L) \geqq Q(L) \quad \text{if } L > 0. \tag{2}$$

In order to prove (1) it is enough to show that any sequence $\{n'\}$ of positive integers contains a subsequence $\{n''\}$ such that $\limsup_{n''\to\infty} Q_{n''}(L)\leqq Q(L)$. Let P_n and P denote the Lebesgue-Stieltjes measures determined by F_n and F respectively. For every n let x_n be chosen such that

$$Q_n(L) \leqq P_n[x_n, x_n + L] + 1/n.$$

Given $\{n'\}$, let $\{n''\}\subset\{n'\}$ be such that $x_{n''}\to$(some) x. If $x=\infty$ or $-\infty$, then by hypothesis, $P_{n''}[x_{n''}, x_{n''}+L]\to 0$ as $n''\to 0$, thus verifying the inequality. If x is finite, then, for arbitrary $\epsilon>0$,

$$\limsup_{n''\to\infty} P_{n''}[x_{n''}, x_{n''} + L] \leqq P[x - \epsilon, x + L + \epsilon] \leqq Q(L + 2\epsilon),$$

and since Q is continuous at L, this inequality establishes (1). To prove (2), let $\epsilon>0$ be arbitrary, and let $0<L_1<L$ be such that $Q(L_1)>Q(L)-\epsilon$ (since $L>0$ is a continuity point of Q). Let x be such that $P[x, x+L_1]>Q(L_1)-\epsilon$ and select $\delta>0$ such that $x-\delta$ and $x+L_1+\delta$ are continuity points of F and $L_1+2\delta<L$. Then

$$\begin{aligned} Q(L) - 2\epsilon < Q(L_1) - \epsilon &< P[x, x + L_1] \\ &\leqq P[x - \delta, x + L_1 + \delta] = \lim_{n\to\infty} P_n[x - \delta, x + L_1 + \delta] \\ &\leqq \liminf_{n\to\infty} Q_n(L_1 + 2\delta) \leqq \liminf_{n\to\infty} Q_n(L), \end{aligned}$$

which, by the arbitrariness of ϵ, proves (2) and concludes the proof of the lemma.

3. **Proof of the theorem.** We first prove that the condition is sufficient. For $L\geqq 0$, let $Q_{i,n}(L)$ denote the concentration function of $\sum_{j=1}^{n} a_{ij}X_j$, let $Q_{i,\infty}(L)$ denote the concentration function of S_i, let

$Q_{\infty,n}(L)$ denote the concentration function of $\sum_{i=1}^{n} X_i$, and let $Q(L)$ denote the concentration function of S. (Note: $L>0$ and S being finite with positive probability imply $Q(L)>0$.) Let $L>0$ be a continuity point of Q, $Q_{\infty,n}$, and $Q_{n,\infty}$, $n=1, 2, \cdots$. Then the lemma and (iii) imply $Q_{i,\infty}(L)\to Q(L)>0$ as $i\to\infty$. For $\epsilon>0$, $0<2\epsilon<Q(L)$, we have

$$Q_{i,\infty}(L) > Q(L) - \epsilon \tag{3}$$

for all $i>$(some) i_0. Take a fixed value of n. Then (i) implies that $\sum_{j=1}^{n} a_{ij}X_j\to\sum_{j=1}^{n} X_j$ a.s. as $i\to\infty$, which in turn implies by the lemma that $Q_{i,n}(L)\to Q_{\infty,n}(L)$ as $i\to\infty$, or

$$Q_{\infty,n}(L) > Q_{i,n}(L) - \epsilon \tag{4}$$

for all $i>$(some) i_1. By Lemma 2 in [4], $Q_{i,n}(L)\geqq Q_{i,\infty}(L)$, and hence by (4), we have

$$Q_{\infty,n}(L) > Q_{i,\infty}(L) - \epsilon \tag{5}$$

for all $i>i_1$. For $i>\max\{i_0, i_1\}$, we obtain, by (3) and (5), the inequality

$$Q_{\infty,n}(L) > Q(L) - 2\epsilon.$$

Since (by Lemma 2 in [4]), $\{Q_{\infty,n}(L)\}$ is a nonincreasing sequence (in n), then

$$\lim_{n\to\infty} Q_{\infty,n}(L) > Q(L) - 2\epsilon > 0. \tag{6}$$

Inequality (6) and the theorem of K. Ito (p. 720 in [4]) then imply the sufficiency of the condition.

We now prove that the condition is necessary. Let $\{\lambda_n\}$ be a sequence of real numbers such that $\sum_{n=1}^{\infty} (X_n-\lambda_n)$ converges a.s.; we shall prove that there exist real numbers $\{a_{ij}\}$ such that (i), (ii) and (iii) are true. We assume that $\sum_{n=1}^{\infty} X_n$ does not converge a.s., i.e., $\sum_{n=1}^{\infty} \lambda_n$ diverges; otherwise $a_{ij}=1$ for all i and j. The proof is broken up into three cases.

Case 1. Suppose the sequence $\{\lambda_n\}$ is unbounded. Then for every positive integer m there is an integer $n_m>m$ such that $\left|\sum_{k=1}^{m} \lambda_k/\lambda_{n_m}\right|\leqq 1$. Now define

$$a_{mj} = \begin{cases} 1 & \text{if } 1 \leqq j \leqq m, \\ -\sum_{k=1}^{m} \lambda_k/\lambda_{n_m} & \text{if } j = n_m, \\ 0 & \text{if } j > m, j \neq n_m. \end{cases}$$

Then

$$S_m = X_1 + \cdots + X_m - \left(\sum_{k=1}^{m} \lambda_k / \lambda_{n_m} \right) X_{n_m}$$

$$= (X_1 - \lambda_1) + \cdots + (X_m - \lambda_m) - \left(\sum_{k=1}^{m} \lambda_k / \lambda_{n_m} \right) (X_{n_m} - \lambda_{n_m}).$$

Since $X_{n_m} - \lambda_{n_m} \to 0$ a.s. as $m \to \infty$, and since its coefficient in the last expression is bounded, then $S_m \to \sum_{n=1}^{\infty} (X_n - \lambda_n)$ a.s. as $m \to \infty$. Thus (ii) and (iii) are shown to hold, and (i) is trivially true.

Case 2. Suppose $\{\lambda_n\}$ are bounded, but suppose there exists an increasing sequence of integers $0 = k_0 < k_1 < \cdots$ such that if $\beta_n = \sum_{j=k_{n-1}+1}^{k_n} \lambda_j$, then the $\{\beta_n\}$ are unbounded. Let us denote $Y_n = \sum_{j=k_{n-1}+1}^{k_n} X_j$. (The Y_n's are independent, but no use will be made of this fact.) Now $\sum_{n=1}^{\infty} (Y_n - \beta_n)$ converges a.s. and the $\{\beta_n\}$ are unbounded. Thus Case 1 applies, and there exist real numbers $\{a'_{mn}\}$ such that $a'_{mn} \to 1$ as $m \to \infty$ for each n, $S'_m = \sum_{n=1}^{\infty} a'_{mn} Y_n$ converges a.s., and $S'_m \to S$ a.s. as $m \to \infty$. Let $a_{mn} = a'_{mr}$ if $k_{r-1} + 1 \leqq n \leqq k_r$. Then the same $S'_m = \sum_{n=1}^{\infty} a_{mn} X_n$ converges a.s. and $S'_m \to S$ a.s.

Case 3. Suppose the $\{\lambda_n\}$ are bounded but that there exists no increasing sequence of integers $\{k_n\}$ such that $\{\beta_n\}$ are unbounded, where β_n is defined in Case 2. We denote $\gamma_n = \lambda_1 + \cdots + \lambda_n$.

We first prove that the $\{\gamma_n\}$ are bounded. Suppose this were not so; suppose $\{\gamma_n\}$ are unbounded above. Then there exists a k_1 such that $\gamma_{k_1} = \lambda_1 + \lambda_2 + \cdots + \lambda_{k_1} = \beta_1 > 1$. In general, there is a k_n such that $\gamma_{k_n} = \lambda_1 + \cdots + \lambda_{k_n} = \beta_n + \gamma_{k_{n-1}} > \gamma_{k_{n-1}} + n$, or $\beta_n > n$. Hence there exists an increasing sequence of integers $\{k_n\}$ such that $\{\beta_n\}$ are unbounded, which contradicts the hypothesis of Case 3. Thus $\{\gamma_n\}$ are bounded.

Since $\{\gamma_n\}$ are bounded, there are limit points of it. There are at least two limit points of $\{\gamma_n\}$, for, otherwise, this sequence would converge, which would contradict our hypothesis. Without loss of generality we may suppose 0 to be the smallest limit point and $a > 0$ the largest. Let $\epsilon > 0$ be such that $3\epsilon < a$. Then there exists an increasing sequence of positive integers $\{k_n\}$ such that $a - \epsilon < \gamma_{k_{2n-1}} < a + \epsilon$ and $-\epsilon < \gamma_{k_{2n}} < \epsilon$ for $n = 1, 2, \cdots$, and $-\epsilon < \gamma_n < a + \epsilon$ for all $n \geqq k_1$. Now let

$$a_{m,1} = a_{m,2} = \cdots = a_{m,k_m} = 1,$$

$$a_{m,k_m+1} = a_{m,k_m+2} = \cdots = a_{m,k_{m+1}} = K_m, \qquad a_{m,j} = 0 \text{ if } j > k_{m+1}$$

where

$$K_m = -\sum_{j=1}^{k_m} \lambda_j \Big/ \sum_{j=k_m+1}^{k_{m+1}} \lambda_j.$$

Then

$$\begin{aligned} S_m &= \sum_{j=1}^{k_m} X_j + K_m \sum_{j=k_m+1}^{k_{m+1}} X_j \\ &= \sum_{j=1}^{k_m} (X_j - \lambda_j) + K_m \sum_{j=k_m+1}^{k_{m+1}} (X_j - \lambda_j). \end{aligned}$$

Since $\{K_m\}$ is bounded, i.e.,

$$|K_m| \leqq \left(\sum_{j=1}^{k_1} \lambda_j + 2\epsilon + a \right) \Big/ (a - 2\epsilon),$$

it follows that $S_m \to \sum_{n=1}^{\infty} (X_n - \lambda_n)$ a.s. as $m \to \infty$, which concludes the proof that the condition is necessary.

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