ON THE COMMUTATIVITY OF RESTRICTED LIE ALGEBRA

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The purpose of this note is to prove the following.

THEOREM. *A finite-dimensional restricted Lie algebra L over an algebrically closed field F with characteristic $p \neq 0$ is abelian whenever the p-map has no nontrivial zero.*

PROOF. Let ϕ denote the p-map in L. For $x \in L$, let V_x be the subspace of L that is spanned by $x, \phi(x), \phi^2(x), \cdots$. Clearly, $V_{\phi(x)} \subset V_x$. Also V_x has a basis consisting of elements of the form $\phi^k(x)$. The images under ϕ of these basis elements are linearly independent elements of $V_{\phi(x)}$, for if $\sum_k a_k\phi^{k+1}(x) = 0$, then

$$\phi\left(\sum_k (a_k)^{1/p}\phi^k(x)\right) = 0,$$

and so

$$\sum_k (a_k)^{1/p}\phi^k(x) = 0.$$

Hence the dimension of $V_{\phi(x)}$ is at least as large as the dimension of V_x, whence $V_{\phi(x)} = V_x$. It follows that, for every positive integer k, $V_x = V_{\phi^k(x)}$. In particular, $x \in V_{\phi^k(x)}$. Let D_x denote the inner derivation of L that is effected by x. Then it follows immediately from our result that, for every $y \in L$ and every positive integer n, $D_x^n(y) = 0$ implies $[x, y] = 0$.

Now suppose that $[x, y] \neq 0$. Let W be the subspace of L that is spanned by $y, D_x(y), D_x^2(y), \cdots$. Since $D_x^n(y) \neq 0$ for any n, D_x is not nilpotent on W. Hence there is a nonzero subspace W_1 of W on which D_x induces a linear automorphism. If we take W_1 of minimal dimension then it is irreducible and hence, by Schur's Lemma, 1-dimensional. Thus there is a nonzero element z in W such that $D_x(z) = az$, where a is a nonzero element of F. This may be written $D_z(x) = -az$ and gives $D_z^2(x) = 0$, whence $D_z(x) = 0$, which contradicts $az = 0$. Thus $[x, y] = 0$, and we have shown that L is abelian.

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