HOMOTOPY FOR CELLULAR SET-VALUED FUNCTIONS

T. R. BRAHANA,[1] M. K. FORT, JR.,[2] AND WALT G. HORSTMAN[3]

1. **Introduction.** A. Granas asked the following question. If F is an upper semi-continuous set-valued function on a compact metric space M such that the image of each point of M is a proper subcontinuum of S^n, then is F "homotopic" to a single-valued continuous function? It was pointed out that care must be used in the definition of homotopy of set-valued functions, since the first natural candidate puts all upper semi-continuous set-valued functions into one class. In [3] and [2] studies were made of homotopies of set-valued functions subject to the restriction that $H(x, t)$ be acyclic (with respect to homology over Z_2) for each $(x, t) \in M \times I$.

In this paper the homotopy problem is solved for those upper semi-continuous functions F for which each $F(x)$ is a cellular subset of S^n. In particular, the class of cellular upper semi-continuous set-valued functions is partitioned into equivalence classes by the relation of cellular homotopy, each class contains single-valued continuous functions, and two single-valued continuous functions are in the same class if and only if they are homotopic in the usual sense.

A selection theorem which seems to be different from those discussed in the literature is proved in §2. It is shown that if F is upper semi-continuous on M and $F(x)$ is a cellular subset of S^n for each $x \in M$, then there exists a continuous function $g: M \to S^n$ such that $g(x) \in S^n - F(x)$ for each $x \in M$. In addition to being the main tool used in the construction of the homotopies, this selection theorem is of interest in itself.

2. **The selection theorem.** A subset A of S^n is *cellular* if and only if there exists a sequence $E_1 \supset E_2 \supset E_3 \supset \cdots$ of topological n-cells such that $A = \bigcap_{k=1}^{\infty} E_k$ and, for each k, $A \subset$ interior E_k.

Let M be an m-dimensional compact metric space and let F be a set-valued function on M such that:

(i) for each $x \in M$, $F(x)$ is a cellular subset of S^n, and

(ii) F is upper semi-continuous.

A covering pair for F and M is an ordered pair (G, D) such that:

Received by the editors December 23, 1963.

[1] Supported by the National Science Foundation research grant G-10096.

[2] Professor Fort died on August 2, 1964. Research had been supported by the National Science Foundation research grant G-23790.

[3] Supported by the National Science Foundation research grant G-10096.

(i) G is a finite open covering of M,

(ii) D is a function with domain G such that for each $U \in G$, $D(U)$ is a topological n-cell which is contained in S^n, and

(iii) for each $x \in M$, if $x \in U \in G$ then $F(x) \subset$ interior $D(U)$.

LEMMA 1. *There exists a covering pair.*

PROOF. For each $x \in M$, there exists a topological n-cell $\Delta(x)$ such that $F(x) \subset$ interior $\Delta(x)$ and $\Delta(x) \subset S^n$. For each $x \in M$, there exists a neighborhood $V(x)$ such that if $t \in V(x)$ then $F(t) \subset$ interior $\Delta(x)$. $\{V(x) \mid x \in M\}$ is an open covering of M, and, since M is compact, this covering has a finite subcovering $G = \{V(x_1), \cdots, V(x_k)\}$. We define $D(V(x_j)) = \Delta(x_j)$ for $j = 1, \cdots, k$. It is easy to verify that (G, D) is a covering pair.

LEMMA 2. *If (G, D) is a covering pair, then there exists a covering pair (G^*, D^*) such that:*

(i) *G^* is a star refinement of G, and*

(ii) *if $U \in G$, $U^* \in G^*$ and $U^* \subset U$, then $D^*(U^*) \subset D(U)$.*

PROOF. Let λ be the Lebesgue number of the covering G. For each $x \in M$, there is a topological n-cell $\Delta(x)$ such that $F(x) \subset$ interior $\Delta(x)$ and such that if $x \in U \in G$ then $\Delta(x) \subset D(U)$. For each $x \in M$, there is a neighborhood $W(x)$ of x such that:

(i) $W(x)$ is contained in the $\lambda/3$-neighborhood of x, and

(ii) if $t \in W(x)$ then $F(t) \subset$ interior $\Delta(x)$. The set $\{W(x) \mid x \in M\}$ is an open covering of M and has a finite subcover $G^* = \{W(x_1), \cdots, W(x_k)\}$. We define $D^*(W(x_j)) = \Delta(x_j)$ for $j = 1, \cdots, k$. It is easy to verify that (G^*, D^*) has the desired properties.

Let F be an upper semi-continuous set-valued function on a compact finite-dimensional metric space M such that, for each $x \in M$, $F(x)$ is a cellular subset of S^n.

THEOREM 1. *There exists a single-valued continuous function $g: M \to S^n$ such that $g(x) \in S^n - F(x)$ for each $x \in M$.*

PROOF. Let m be the dimension of M. It follows from Lemmas 1 and 2 that there are covering pairs $(G_0, D_0), \cdots, (G_{2m}, D_{2m})$ such that for each j, $1 \leq j \leq 2m$: (i) G_j is a star refinement of G_{j-1}, and (ii) if $U_j \in G_j$, $U_{j-1} \in G_{j-1}$ and $U_j \subset U_{j-1}$, then $D_j(U_j) \subset D_{j-1}(U_{j-1})$.

We choose a finite open covering G of M such that G is of order m, G is a star refinement of G_{2m}, and no proper subset of G covers M. For each integer j, $0 \leq j \leq m$, we define $K_j = \{x \mid x \in M$ and x is a member of at most $j+1$ members of $G\}$. Each K_j is a closed subset of M, and $K_m = M$.

For each $V\in G$ and each integer j, $0\leqq j\leqq 2m$, we select sets $\phi_j(V)\in G_j$ such that $\mathrm{St}(V)\subset\phi_{2m}(V)$ and $\mathrm{St}(\phi_j(V))\subset\phi_{j-1}(V)$, for $1\leqq j\leqq 2m$. We define $\Phi_j(V)=D_j(\phi_j(V))$ for each $V\in G$ and $j=0,\cdots,2m$.

We are going to define (inductively) for each j, $0\leqq j\leqq m$, a mapping $g_j\colon K_j\to S^n$ such that, for each $V\in G$,

$$g_j[V\cap K_j]\subset \mathrm{Cl}[S^n-\Phi_{2j}(V)].$$

For each $V\in G$, we choose a point $p_V\in S^n-\Phi_0(V)$ and define $g_0(x)=p_V$ for each $x\in V\cap K_0$. Since $V\cap K_0$ is closed in K_0 for each $V\in G$, g_0 is continuous.

Now suppose $0<j\leqq m$ and g_{j-1} has been defined. Let $\sigma=\{V_0,\cdots,V_j\}$ be a set of $j+1$ distinct members of G such that $V_0\cap\cdots\cap V_j\neq\emptyset$. We define $H_\sigma=(V_0\cap\cdots\cap V_j)\cap K_j$ and $W_\sigma=K_j-\bigcup\{V\mid V\in G-\sigma\}$. Then W_σ is closed in K_j and H_σ is open relative to W_σ. The mapping g_{j-1} is defined on $W_\sigma-H_\sigma$ and $W_\sigma-H_\sigma$ is closed relative to W_σ. It is easy to see that

$$\begin{aligned} g_{j-1}[W_\sigma-H_\sigma] &\subset \bigcup_{r=0}^{j}\mathrm{Cl}[S^n-\Phi_{2j-2}(V_r)] \\ &= \mathrm{Cl}\left[S^n-\bigcap_{r=0}^{j}\Phi_{2j-2}(V_r)\right]. \end{aligned}$$

Since $\phi_{2j-1}(V)\subset\mathrm{St}(\phi_{2j-1}(V_r))\subset\phi_{2j-2}(V_r)$ for $r=0,\cdots,j$, $\Phi_{2j-1}(V_0)\subset\bigcap_{r=0}^{j}\Phi_{2j-2}(V_r)$. Therefore, $g_{j-1}[W_\sigma-H_\sigma]\subset\mathrm{Cl}[S^n-\Phi_{2j-1}(V_0)]$.

The set $\mathrm{Cl}[S^n-\Phi_{2j-1}(V_0)]$ is the union of a topological $(n-1)$-sphere Σ and one of the components of $S^n-\Sigma$. It is known (see [1]) that such sets are absolute retracts. Since $\mathrm{Cl}[S^n-\Phi_{2j-1}(V_0)]$ is an absolute retract, we can extend $g_{j-1}\mid(W_\sigma-H_\sigma)$ to a mapping

$$\psi_\sigma\colon W_\sigma\to\mathrm{Cl}[S^n-\Phi_{2j-1}(V_0)].$$

Since $V_r\subset\mathrm{St}(V_0)$ for $r=0,\cdots,j$,

$$\phi_{2j}(V_r)\subset\mathrm{St}(\phi_{2j}(V_0))\subset\phi_{2j-1}(V_0).$$

Thus $\Phi_{2j}(V_r)\subset\Phi_{2j-1}(V_0)$ and the range of ψ_σ is contained in $\mathrm{Cl}[S^n-\Phi_{2j}(V_r)]$ for $r=0,\cdots,j$. Thus $\psi_\sigma[V_r\cap W_\sigma]\subset\mathrm{Cl}[S^n-\Phi_{2j}(V_r)]$ for $r=0,\cdots,j$.

If σ' is a different system of $j+1$ members of G and $W_{\sigma'}\cap W_\sigma\neq\emptyset$, then $\psi_\sigma\mid(W_{\sigma'}\cap W_\sigma)=\psi_{\sigma'}\mid(W_{\sigma'}\cap W_\sigma)=g_{j-1}\mid(W_{\sigma'}\cap W_\sigma)$. It follows that we can piece the mappings ψ_σ and g_{j-1} together to obtain a mapping $g_j\colon K_j\to S^n$. It is obvious that $g_j[V\cap K_j]\subset\mathrm{Cl}[S^n-\Phi_{2j}(V)]$ for each $V\in G$.

We define $g=g_m$. Since $K_m=M$, g is a mapping of M into S^n such that $g[V] \subset \mathrm{Cl}[S^n - \Phi_{2m}(V)]$ for each $V \in G$. If $x \in V \in G$, then $x \in \phi_{2m}(V)$ and, hence, $F(x) \subset$ interior $\Phi_{2m}(V)$. It follows that if $x \in M$, then $g(x) \in S^n - F(x)$.

3. **Homotopy for a class of set-valued functions.** Let M be a finite-dimensional compact metric space. We define $\Gamma(M, S^n)$ to be the set of all upper semi-continuous set-valued functions F on M such that for each $x \in M$, $F(x)$ is a cellular subset of S^n. We let $I = [0, 1]$.

Let F and G be members of $\Gamma(M, S^n)$. A function H is a *cellular homotopy* relating F to G if:

(i) $H \in \Gamma(M \times I, S^n)$, and

(ii) for all $x \in M$, $H(x, 0) = F(x)$ and $H(x, 1) = G(x)$.

If there exists a cellular homotopy relating F to G, then we say that F is *homotopic* to G and write $F \sim G$. The relation $\sim$ is an equivalence relation and partitions $\Gamma(M, S^n)$ into equivalence classes which we call *cellular homotopy classes.*

Let $F \in \Gamma(M, S^n)$ and let $f: M \to S^n$ be a (single-valued) continuous function. A function H is a *special homotopy* relating F to f if:

(i) H is a cellular homotopy relating F to f, and

(ii) for all $x \in M$ and $0 \leqq t < 1$, $H(x, t)$ is homeomorphic to $F(x)$.

If F is a single-valued function as well as f, then (ii) implies that H is single-valued, and since upper semi-continuity is equivalent to continuity for single-valued functions, in this case H is an ordinary homotopy.

LEMMA 3. *If $F \in \Gamma(M, S^n)$, then there exists a single-valued continuous function $f: M \to S^n$ and a special homotopy H relating F to f.*

PROOF. For each $p \in S^n$, we define a mapping $J_p: [S^n - p] \times I \to S^n$ by

$$J_p(x, t) = [-tp + (1 - t)x]/\| - tp + (1 - t)x\|$$

for $x \in S^n - p$, $0 \leqq t \leqq 1$. J_p is a pseudo-isotopy, since the map ϕ_t defined by $\phi_t(x) = J_p(x, t)$ is a homeomorphism on $S^n - p$ if $0 \leqq t < 1$, ϕ_0 is the identity mapping on $S^n - p$, and ϕ_1 is the constant map which takes $S^n - p$ into $-p$.

By Theorem 1, there is a mapping $g: M \to S^n$ such that $g(x) \in S^n - F(x)$ for each $x \in M$. We define $f(x) = -g(x)$ and

$$H(x, t) = \{J_{g(x)}(y, t) \mid y \in F(x)\}$$

for $x \in M$ and $0 \leqq t \leqq 1$. Obviously, f is a continuous function on M into S^n, and it is easy to verify that H is a special homotopy relating F to f.

THEOREM 2. *Each cellular homotopy class of* $\Gamma(M, S^n)$ *contains a single-valued continuous function* $f: M \to S^n$.

PROOF. This result follows immediately from Lemma 3 and the fact that special homotopies are cellular homotopies.

Our final theorem shows that the notion of cellular homotopy which we have defined for $\Gamma(M, S^n)$ is a true extension of the usual notion of homotopy for single-valued functions.

THEOREM 3. *If* f_0 *and* f_1 *are single-valued continuous functions on* M *into* S^n *and* $H \in \Gamma(M \times I, S^n)$ *is a cellular homotopy relating* f_0 *to* f_1, *then there exists a single-valued homotopy* $h: M \times I \to S^n$ *which relates* f_0 *to* f_1 *in the usual sense.*

PROOF. We apply Lemma 3 (replacing M by $M \times I$ and F by H) to obtain a single-valued continuous function $\phi: M \times I \to S^n$ and a special homotopy $K \in \Gamma((M \times I) \times I, S^n)$ relating H to ϕ. Now, for $(x, t) \in M \times I$, we define

$$h(x, t) = \begin{cases} K(x, 0, 3t) & \text{if } 0 \leqq t \leqq 1/3, \\ K(x, 3t - 1, 1) & \text{if } 1/3 \leqq t \leqq 2/3, \\ K(x, 1, 3 - 3t) & \text{if } 2/3 \leqq t \leqq 1. \end{cases}$$

If $0 \leqq t < 1/3$, then $h(x, t) = K(x, 0, 3t)$ is homeomorphic to $K(x, 0, 0) = H(x, 0) = f_0(x)$ and, hence, is a one-point set. Likewise, if $2/3 < t \leqq 1$, then $h(x, t)$ is a one-point set. If $1/3 \leqq t \leqq 2/3$, then $h(x, t) = K(x, 3t-1, 1) = (x, 3t-1)$ and since ϕ is single-valued, $h(x, t)$ is a one-point set. Thus h is a single-valued function on $M \times I$ into S^n. Since K is upper semi-continuous, h is also upper semi-continuous. Since h is single-valued, this implies that h is continuous.

We have shown that $h: M \times I \to S^n$ is an ordinary single-valued homotopy. Since $h(x, 0) = K(x, 0, 0) = H(x, 0) = f_0(x)$ and $h(x, 1) = K(x, 1, 0) = H(x, 1) = f_1(x)$, h relates f_0 to f_1 in the usual sense.

It should be remarked that it can be shown that the smallest equivalence relation containing both homotopies of single-valued functions and special homotopies is the relation generated by cellular homotopies. The proof is similar to that of Theorem 4.

BIBLIOGRAPHY

1. R. H. Bing, *Retractions onto spheres*, Amer. Math. Monthly **71** (1964), 481–484.

2. A. Granas and J. W. Jaworowski, *Some theorems on multi-valued mappings of subsets of the Euclidean space*, Bull. Acad. Polon. Sci. Sér. **7** (1959), 277–283.

3. J. W. Jaworowski, *Some consequences of the Vietoris mapping theorem*, Fund. Math. **45** (1958), 261–272.

UNIVERSITY OF GEORGIA