A GEOMETRIC CHARACTERIZATION OF GLEASON PARTS

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1. **Introduction.** In [3] Gleason introduced an equivalence relation on the maximal ideal space S_A of a function algebra A, which divides S_A into equivalence classes called "parts", or "Gleason parts". The decomposition of S_A into these parts is an important step in the attempt to find whatever analytic structure S_A may have (cf. Wermer's elegant result in [7]). Among other properties of parts, Gleason showed that any two points in the same part of a Dirichlet algebra have representing measures on the Silov boundary which are mutually absolutely continuous. Bishop [1] has recently given a simple proof that this fact holds for any function algebra. Motivated by the ideas of Bishop's proof, we wish to extend the notion of Gleason part to an arbitrary linear space B of continuous real functions on a compact space X. The parts of X determined by B are then subsets of X on which a Harnack inequality holds, in much the same way that the Gleason parts for an algebra on X are subsets of X on which a type of Schwarz lemma holds. The equivalence relation we introduce is a generalization of Gleason's, in the sense that if A is an algebra on X, and B is the space of real parts, then the classes of X induced by B coincide with the Gleason parts. We also introduce a metric on the parts of X induced by B, and show that, for the case $B = \text{Re } A$, it is equivalent to the norm metric studied by Gleason. Finally, the parts are characterized geometrically as the minimal faces of a certain compact convex subset of the dual space.

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2. **The parts of X induced by B.** Let X be a compact Hausdorff space, and let $C_r(X)$ (respectively, $C_c(X)$) be the space of all continuous real-valued (respectively, complex-valued) functions on X, with the uniform norm. We consider a linear subspace B of $C_r(X)$, containing the constant functions, and separating the points of X.

For any two points $x, y \in X$ write $x \sim y$ (a) if and only if

$$1/a < u(x)/u(y) < a \tag{1}$$

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for all strictly positive functions $u \in B$. The condition (1) can be viewed as an abstract form of Harnack's inequality for positive harmonic functions. If $x \sim y$ (a) then, clearly, $a > 1$. We write $x \sim y$ if and only if $x \sim y$ (a) for some a. It is easy to check that $\sim$ is an equivalence relation on X and we state the relevant facts in the following lemma.

LEMMA 1. (i) *$x \sim x$ (a) for all $a > 1$*; (ii) *$x \sim y$ (a) implies $y \sim x$ (a)*; (iii) *$x \sim y$ (a) and $y \sim z$ (b) implies $x \sim z$ (ab).*

The equivalence classes of X under $\sim$ will be called the "parts", or "Gleason parts" of X induced by B. For x and y in the same part, we define

$$R(x, y) = \inf\{a : x \sim y\ (a)\}. \tag{2}$$

The following is immediate from Lemma 1 and this definition.

LEMMA 2. *If x, y, z are in the same part of X, then* (i) $R(x, y) \geqq 1$ *and* $R(x, y) = 1$ *if and only if* $x = y$; (ii) $R(x, y) = R(y, x)$; (iii) $R(x, y)R(y, z) \geqq R(x, z)$.

Let D be the function on all pairs (x, y) of points in the same part defined by

$$D(x, y) = \log R(x, y). \tag{3}$$

LEMMA 3. *D is a metric on each part of X.*

3. **Relationship with the Gleason parts of an algebra.** Let A be a closed subalgebra of $C_c(X)$, and assume that A contains the constant functions and separates the points of X. We will say "A is an algebra on X" if these conditions hold. If A^* is given the w^* topology, then X is homeomorphic to the subset of A^* consisting of the evaluation functionals e_x defined by $e_x(f) = f(x)$ for all $f \in A$. We can transfer the norm-metric of A^* to X via this identification, and we write, for $x, y \in X$,

$$G(x, y) = \|e_x - e_y\| = \sup\{|f(x) - f(y)| : f \in A, \|f\| < 1\}. \tag{4}$$

Clearly G is a metric, and $G(x, y) \leqq 2$ for all x, y.

THEOREM 4 (GLEASON [3]). *If $G(x, y) < 2$ and $G(y, z) < 2$, then $G(x, z) < 2$.*

Theorem 4 implies that the relation $G(x, y) < 2$ is an equivalence relation on X, which we denote $x \approx y$. We will need the following fact, which follows from the properties of conformal mappings of the unit disc, applied to the range of functions in A. The points x, y are not equivalent ($x \approx y$ fails, or $G(x, y) = 2$) if and only if for some $s < 1$ and every $\epsilon > 0$, there is $f \in A$, $\|f\| < 1$, with $|f(x)| \leqq s$ and $|f(y)| > 1 - \epsilon$. The following theorem shows that the relation $\sim$ does specialize to the Gleason relation $\approx$ in the case of a function algebra. The basic idea of this theorem is due to Bishop [1] who showed in somewhat different language that $x \approx y$ implies $x \sim y$.

THEOREM 5. *If A is an algebra on X and $B = \operatorname{Re} A$, then $G(x, y) < 2$ if and only if $1/a < u(x)/u(y) < a$ for some a and all positive $u \in B$.*

We can state the theorem briefly: $x \sim y$ if and only if $x \approx y$.

PROOF. Assume that condition (1) does not hold; that is, $x \sim y$ fails. If, to be specific, $u(x)/u(y)$ is not bounded away from zero, then there is a strictly positive $u \in B$ such that $u(x) < \epsilon$ and $u(y) = 1$. If $u + iv \in A$ and $f = e^{-u-iv}$, then $f \in A$, $|f| = e^{-u} < 1$, $|f(y)| = e^{-1}$, and $|f(x)| > e^{-\epsilon}$. Thus there are functions f in A of norm $\leqq 1$ which map x arbitrarily close to the boundary of the disc, while $|f(y)|$ is bounded away from 1. This shows that $G(x, y) = 2$ if $x \sim y$ fails.

Now assume $G(x, y) = 2$. Given $\epsilon > 0$ there is therefore $f = u + iv \in A$ such that $\|f\| < 1$ and $u(x) < -1 + \epsilon$, $u(y) > 1 - \epsilon$. If $u_0 = u + 1$, then $u_0 \in B$, $u_0 > 0$ on X, and $u_0(x)/u_0(y) < \epsilon/(2 - \epsilon)$. Hence $x \sim y$ fails if $x \approx y$ fails.

THEOREM 6. *If A is an algebra on X, and $B = \operatorname{Re} A$, then the metric G of* (4) *is equivalent to the metric D of* (3) *on each part of X.*

PROOF. Fix a part of X, and consider only points x, y in this part. We show that if $G(x, y) \geqq r > 0$, then $D(x, y) \geqq \log(1 + r/2)$, and secondly, that if $D(x, y) \geqq r > 0$, then $G(x, y) \geqq e^{-1} - e^{-1-r}$.

Assume first that $G(x, y) \geqq r > 0$. For any $\epsilon > 0$ there is $f \in A$, $\|f\| < 1$, such that $|f(x) - f(y)| > r - \epsilon$. We can assume, by rotating the graph of f if necessary, that $f = u + iv$ and $u(x) - u(y) > r - \epsilon$. If $u_0 = u + 1$, then $0 < u_0 < 2$ on X, $u_0(x) - u_0(y) > r - \epsilon$, and

$$u_0(x)/u_0(y) > (u_0(y) + r - \epsilon)/u_0(y) > 1 + (r - \epsilon)/2.$$

Hence $R(x, y) \geqq 1 + r/2$, and $D(x, y) \geqq \log(1 + r/2)$.

Now assume that $D(x, y) = \log R(x, y) \geqq r > 0$. Since $R(x, y) \geqq e^r > 1 + r$, there is a positive function $u \in B$ with $u(x)/u(y) > 1 + r$ (or $u(y)/u(x) > 1 + r$ with a similar proof). We normalize u so that $u(y) = 1$ and $u(x) > 1 + r$. Let $u + iv \in A$, and $f = e^{-u-iv}$. Then $f \in A$, $\|f\| < 1$, and

$$|f(x) - f(y)| \geqq |f(y)| - |f(x)| = e^{-u(y)} - e^{-u(x)} > e^{-1} - e^{-1-r}.$$

COROLLARY. *Each $f \in A$ is continuous on each part of X with the metric G or D.*

PROOF. This is obvious for the metric G, since the norm topology in A^* is stronger than the w^* topology, which coincides on the parts with the given topology. Hence the functions are also continuous in the equivalent metric D.

4. **Geometric characterization of parts.** Our geometric characterization of the Gleason parts for a subspace B of $C_r(X)$ (and hence of a subalgebra A of $C_c(X)$) makes use of the representation of B as the weak dual of B^*. Give B^* the w^* topology, and, for each $u \in B$, define $\bar{u}$ on B^* by $\bar{u}(F) = F(u)$ for all $F \in B^*$. The space $\overline{B}$ of all such functions $\bar{u}$ is isomorphic to B, and is the space of all w^*-continuous linear functionals on B^* ([2, p. 18] or [4, Problem W, p. 108]). If $X^e = \{e_x : x \in X\} \subset B^*$, then X^e is homeomorphic to X and $\overline{B} | X^e$ is a copy of B on X, since $\bar{u}(e_x) = e_x(u) = u(x)$. Let T_B denote the closed convex hull of X^e in B^*. The following theorem can be derived as a consequence of the Riesz representation theorem and the fact that linear combinations of point masses are w^* dense in the space of all measures. We include a simple proof which does not involve any notion of measure or integration.

THEOREM 7. $T_B = \langle X^e \rangle = \{F \in B^* : F(1) = \|F\| = 1\}$.

PROOF. Since B^* is a locally convex space, and $\overline{B}$ is its dual, the closed convex hull of X^e can be written [2, Corollary 2, p. 22]

$$
\begin{aligned}
T_B = \langle X^e \rangle &= \{F \in B^* : \bar{u}(F) \leqq \max \bar{u}[X^e], \quad \text{all } u \in B\} \\
&= \{F \in B^* : F(u) \leqq \max u, \quad \text{all } u \in B\}.
\end{aligned}
\tag{5}
$$

For $F \in T_B$ we have, from (5), $F(u) \leqq \max u \leqq \|u\|$ and $-F(u) \leqq F(-u) \leqq \max(-u) \leqq \|u\|$. Hence $-\|u\| \leqq F(u) \leqq \|u\|$, and $\|F\| \leqq 1$. In addition, $-F(1) = F(-1) \leqq \max(-1) = -1$, so $F(1) = 1$, and $\|F\| = F(1) = 1$ if $F \in T_B$. Now suppose that $\|F\| = F(1) = 1$. If $F \notin T_B$, then $F(u_0) > \max u_0$ for some $u_0 \in B$. For sufficiently large n, $u_0 + n \geqq 0$, and $\|n + u_0\| = n + \max u_0$. Since $F(1) = 1$, $F(n + u_0) = n + F(u_0) > \|n + u_0\|$, which contradicts the assumption $\|F\| = 1$.

The set T_B, which we will call the *carrier* of B, is a compact convex set in the locally convex space B^*, w^*. It is the largest compact space K containing X such that B can be extended to K to be isomorphic and isometric with a separating subspace of $C_r(K)$ containing the constants. In this sense, the carrier T_B plays somewhat the role for a linear space B that the spectrum S_A does for an algebra A. If A is an algebra on X, and $B = \operatorname{Re} A$, then S_A can be considered a subset of T_B, since the homomorphisms of A are *positive* linear functionals of norm one (cf. [6, p. 58]). We can therefore consider A as *linearly* isomorphic and isometric to a linear space $\overline{A}$ of complex linear functionals on T_B, with $\overline{A}|S_A$ being *algebraically* isomorphic to A.

To simplify the notation, and without loss of generality, we will henceforth regard X as a compact subset of a locally convex space, B^*, with compact closed convex hull T_B, and B as the space of all continuous linear functionals on B^*, usually restricted to X or to T_B. We use x, y, z for points of X, or T_B, or functionals in B^*.

We will call a subset E of a convex set K a *face* of K if and only if E is convex and $(a, b) \subset E$ whenever the segment (a, b) intersects E and $(a, b) \subset K$. Note that a face differs from the "extremal set" of [2, p. 78] and the "support set" of [5, p. 130] in that a face should not contain all end points of segments $[a, b]$ which intersect it. A face is "open" in this sense. A nonempty intersection of faces is obviously a face. We will let $S(x)$ denote the smallest face of T_B containing x; i.e., $S(x)$ is the intersection of all faces of T_B containing x.

LEMMA 8. *Either x is an extreme point of T_B, or $S(x)$ is the union of all open segments (a', a'') in T_B which contain x.*

PROOF. Suppose x is not an extreme point of T_B. Clearly $S(x)$ must contain all open segments (a', a'') such that $x \in (a', a'') \subset T_B$, so it is sufficient to show that the union E of such segments is a face. If $a, b \in E$, then $a \in (a', a'') \subset T_B$, and $b \in (b', b'') \subset T_B$, and x is in both segments. The points $a, a', a'', b, b', b'', x$ all lie in a two-dimensional plane in T_B, and it is easy to see from the plane geometry that each point of (a, b) is in an open segment of T_B containing x. Thus E is convex. It similarly is clear from the two-dimensional picture that if $y \in E$ and $y \in (a, b) \subset T_B$, that $(a, b) \subset E$.

It follows from the definition of a face that if $y \in S(x)$, then $S(y) \subset S(x)$. On the other hand, from Lemma 8 we have that $y \in S(x)$ if and only if $x \in S(y)$. Thus $y \in S(x)$ if and only if $S(x) = S(y)$.

We call attention to the fact that Lemma 8 shows that *any* convex set K decomposes into disjoint convex sets which are its minimal faces, and that each face F (other than an extreme point) can be characterized as the union of all open segments in K through any point of F.

THEOREM 9. *If x and y are points of T_B, then $x \sim y$ if and only if $S(x) = S(y)$. That is, the Gleason parts of T_B are its minimal faces, and the Gleason parts of X are the intersections of X (properly X^e) with the minimal faces of T_B.*

PROOF. Assume that $S(x) \neq S(y)$, and to be definite assume that x is not interior to any segment in T_B containing y. Let z_n be the point of B^* such that

$$x = \frac{n-1}{n} z_n + \frac{1}{n} y. \tag{6}$$

Then $z_n \notin T_B$ by hypothesis. However, z_n is in the hyperplane $\{F \in B^*: F(1) = 1\}$, since $x(1) = 1$ and $y(1) = 1$. Let $u \in B$, and $u(z_n) < \inf u[T_B] = a$ (such u exists by [2, Theorem 5, p. 22]). If $u_0 = u - a \in B$, then $u_0(z_n) = u(z_n) - a < 0 = \inf u_0[T_B]$. From (6) we have $u_0(x) < (1/n)u_0(y)$. Since n is arbitrary, $x \sim y$ fails if $S(x) \neq S(y)$. Now assume that x and y are not in the same part, and, to be definite, that $u(x)/u(y)$ is not bounded away from zero for $u > 0$ on T_B. Then there is no $z \in T_B$ such that $x \in (y, z)$, for, if $x = \lambda y + (1-\lambda)z$, and $z \in T_B$, then $u(x)/u(y) \geqq \lambda$ for all positive $u \in B$. From the preceding lemma we conclude that $x \notin S(y)$ and hence that $S(x) \neq S(y)$.

Let us make explicit what the theorem above says for a function algebra A. Suppose that A is an algebra on X, which may be assumed to be the Silov boundary of A, the whole maximal ideal space S_A, or any closed set in between. If $B = \operatorname{Re} A$, then X is (homeomorphic to) a subset of T_B, and A can be considered a family of complex linear functionals on T_B, with A closed under multiplication when restricted to X. The Gleason parts of X with respect to A are the intersections of X with the minimal faces of T_B. Using the characterization of T_B in Theorem 7, we can alternatively say that each part of S_A consists of the multiplicative functionals in one of the minimal faces of the set $\{F \in B^*: F(1) = \|F\| = 1\}$.

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