A LOCALLY COMPACT METRIC SPACE IS ALMOST INVARIANT UNDER A CLOSED MAPPING

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1. **Introduction.** For a given mapping (continuous transformation) of a topological space X onto a topological space Y, it has always been of interest to determine what properties of X carry over to Y. For example, if f is a closed mapping, then normality [1] and paracompactness [5] are invariants. If X is a metric space and f is a closed mapping, then, as a consequence of results by Vainstein [2], Whyburn [1], and Stone [4], it is known that Y is first countable if and only if each point inverse has a compact frontier. A set S of a space X is said to be a *scattered set* if every subset of S is a closed subset of X. The main result of this paper shows that the image Y, of a locally compact metric space X under a closed mapping f, minus a scattered set S is also a locally compact metric space. In addition, if X is separable, then S has at most a countable number of points. In case the mapping is also an open mapping, the results are stated in [3], [4], [6] and [7].

2. **Notation.** For a metric space, ρ is used as the distance function. The frontier of a set A in a metric space X is the closure minus its interior relative to X and is denoted by Fr A. An inverse set of a mapping f of X onto Y is any subset A of X for which $A=f^{-1}f(A)$. Other undefined terms can be extracted from [1], [4] and [5].

3. **Main result.**

THEOREM. *Let f be a closed mapping of a locally compact metric space X onto a topological space Y. If S is the set of all y in Y such that the frontier of $f^{-1}(y)$ is not compact, then S is a scattered set and $Y-S$ is a locally compact metric space. Moreover, if X is separable, then S is countable.*

To facilitate the proof of the main theorem, the following lemma is established.

LEMMA. *Let f be a closed mapping of X onto Y, where X is a locally compact metric space. If F is the union of the point inverses which are not compact, then*

(i) *F is a closed set*;

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(ii) *for an arbitrary compact set K in X, only a finite number of non-compact point inverses can intersect K;*
(iii) *any inverse set A contained in F is a closed subset of X.*

PROOF. If there are no compact point inverses, then $F=X$ and F is closed. Suppose that $f^{-1}(y)$ is a compact point inverse. Let U be an open set containing $f^{-1}(y)$ and such that $\overline{U}$ is compact. The set $f^{-1}f(X-U)$ is closed and its complement is an open inverse set V containing $f^{-1}(y)$ and contained in U. Each compact point inverse is contained in such a set so that $X-F$ is an open set and consequently F is a closed set.

To prove (ii), suppose K is a compact set in X and $f^{-1}(y_n)$ is a sequence of distinct noncompact point inverses, each of which intersects K in a nonempty set. We can assume that $\liminf f^{-1}(y_n) \neq \emptyset$ and, hence, if $L=\limsup f^{-1}(y_n)$, then L is in some point inverse. Suppose, further, that L is not in any one of the $f^{-1}(y_n)$. If L is a compact set, then there is an open set W containing L with $\overline{W}$ compact. Since no $f^{-1}(y_n)$ is compact, each one has at least one point x_n not in $\overline{W}$. The set $\bigcup_{n=1}^{\infty} x_n$ is a closed set but $f^{-1}f(\bigcup_{n=1}^{\infty} x_n)=\bigcup_{n=1}^{\infty} f^{-1}(y_n)$ is not a closed set. This is contrary to f being a closed map. Thus, L is not compact. If L is not compact, it contains an infinite sequence of distinct points y_n, with $\limsup y_n=\emptyset$. There is an $f^{-1}(y_{n_1})$ and a point x_1 of $f^{-1}(y_{n_1})$ such that $\rho(x_1, y_1)<1$, an $f^{-1}(y_{n_2})$ and a point x_2 of $f^{-1}(y_{n_2})$ such that $\rho(x_2, y_2)<1/2$, and, in general, an $f^{-1}(y_{n_k})$ and a point x_k of $f^{-1}(y_{n_k})$ such that $\rho(x_k, y_k)<1/k$. The sequence x_k must have $\limsup x_k=\emptyset$, otherwise $\limsup y_k \neq \emptyset$. The mapping f is closed and $\bigcup_{k=1}^{\infty} x_k$ is a closed set, hence $f^{-1}f(\bigcup_{k=1}^{\infty} x_k)=\bigcup_{k=1}^{\infty} f^{-1}(y_{n_k})$ is closed. But $\bigcup_{k=1}^{\infty} f^{-1}(y_{n_k})$ must have a limit point in $K \cap L$ and this gives a contradiction.

To prove (iii), let A be an inverse set contained in F. If A is not closed, then there is a sequence x_n of distinct points of A converging to a point x not in A. Since point inverses are closed sets we can assume that $f^{-1}f(x_i) \cap f^{-1}f(x_j)=\emptyset$ if $i \neq j$. The set $K=\bigcup x \bigcup_{n=1}^{\infty} x_n$ is compact and an infinite number of distinct noncompact point inverses meet K. This contradicts (ii) and therefore A is closed.

PROOF OF THE THEOREM. Let B be any subset of S. The set $f^{-1}(B)$ is closed in X by (iii) of the Lemma. The mapping f is closed, hence $ff^{-1}(B)=B$ is closed, and S is a scattered set. The set $Y-S$ is the continuous image of $X-f^{-1}(S)$ under f and the mapping f restricted to $X-f^{-1}(S)$ is a closed mapping of $X-f^{-1}(S)$ onto $Y-S$. By a result of Stone [4] (see Theorem 1), we can say that $Y-S$ is metrizable.

To show $Y-S$ is locally compact, let $y \in Y-S$ and suppose that $\operatorname{Fr} f^{-1}(y) \neq \emptyset$. Let W be any open set containing $\operatorname{Fr} f^{-1}(y)$ and such that $\overline{W}$ is compact. If y is not interior to $f(W)$, then there exists a sequence of points y_n of $(Y-S)-f(W)$ converging to y. Since f is a closed mapping, $\lim\sup f^{-1}(y_n)\subset \operatorname{Fr} f^{-1}(y)$. Thus, infinitely many of the $f^{-1}(y_n)$ meet W, which in turn implies that infinitely many of the y_n are in $f(W)$. This gives a contradiction, hence y is interior to $f(W)$. The interior points of $f(W)$ form an open set whose closure is contained in the compact set $f(\overline{W})$. If $\operatorname{Fr} f^{-1}(y)=\varnothing$, then $\{y\}$ is an open set containing y and $\{\bar{y}\}$ is compact. Thus $Y-S$ is locally compact.

If X is separable, then X is a countable union of compact sets. Only a finite number of point inverses of points in S can meet any one of these compact sets. Thus $f^{-1}(S)$ is a countable union of single point inverses. Therefore S is countable.

4. **Examples.** To see that S in the theorem can be infinite consider the decomposition of the plane E_2 into the vertical lines whose equations are $x=n$, n an integer, and the individual points not on these lines. Let Y be the topological space determined by this upper semi-continuous decomposition of E_2. The natural mapping of E_2 onto Y is a closed mapping and S in this case is denumerable.

The following example shows that if local compactness is dropped in the Lemma, then every conclusion must be omitted. Let X be the open unit disk in the plane union the points $(0, 1)$ and $(0, -1)$. Consider the decomposition of X into the sets determined by intersecting the graphs of $x=1/n$, n a positive integer, and $x=0$, with X, and the individual points not on these lines. Again, let Y be the topological space determined by this upper semi-continuous decomposition of X. The natural mapping of X onto Y is a closed mapping and no one of the conclusions of the Lemma are valid.

Bibliography

1. G. T. Whyburn, *Open and closed mappings*, Duke Math. J. **17** (1950), 69–74·

2. I. A. Vainstein, *On closed mappings in metric spaces*, Dokl. Akad. Nauk SSSR (N.S.) **57** (1947), 319–321. (Russian)

3. A. D. Wallace, *Some characterizations of interior transformations*, Amer. J. Math. **61** (1939), 757–763.

4. A. H. Stone, *Metrizability of decomposition spaces*, Proc. Amer. Math. Soc. **7** (1956), 690–700

5. E. Michael, *Another note on paracompact spaces*, Proc. Amer. Math. Soc. **8** (1957), 822–828.

6. G. T. Whyburn, *Continuous decompositions*, Amer. J. Math. **71** (1949), 218–226.

7. V. K. Balanchandran, *A mapping theorem for metric spaces*, Duke Math. J. **22** (1955), 461–464.

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