ON RESTRICTIONS OF FUNCTIONS IN THE SPACES $P^{\alpha,p}$ AND $B^{\alpha,p}$

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In this note we give a generalization of a result of Aronszajn and Smith [1] concerning sections of exceptional sets for the spaces of Bessel potentials on R^n of L^2 functions and restrictions of functions in these spaces. For the sake of completeness we recall briefly the relevant definitions and theorems. We refer for details to [2]. Throughout this paper we will be concerned with functions defined on the space R^n; we write $\int$ for $\int_{R^n}$, L^p for $L^p(R^n)$, C_0^∞ for $C_0^\infty(R^n)$, etc.

The *Bessel kernel* G_α on R^n of *order* $\alpha>0$ is defined as the inverse Fourier transform of the function

$$\hat{G}_\alpha(\xi) = \frac{(2\pi)^{-n/2}}{(1+|\xi|^2)^{\alpha/2}}. \tag{1}$$

$G_\alpha(x)$ is positive for all $x\in R^n$, $x\neq 0$, also $\|G_\alpha\|_{L^1}=\int G_\alpha(x)\,dx=1$. For $\alpha>0$ and $f\in L^1_{\text{loc}}$ we denote $(G_\alpha f)(x)=(G_\alpha * f)(x)=\int G_\alpha(x-y)f(y)\,dy$, if the integral exists; we also define the operator G_0 as the identity operator.

For $1\leqq p<\infty$ and $\alpha>0$, $\mathfrak{A}_{\alpha,p}$ denotes the class of all sets $A\subset R^n$ for which there exists a function $f\in L^p$, $f\geqq 0$, such that $A\subset\{x: (G_\alpha f)(x)=+\infty\}$. For $\alpha=0$ we define $\mathfrak{A}_{0,p}=\mathfrak{A}_0$ as the class of sets of n-dimensional Lebesgue measure 0. $\mathfrak{A}_{\alpha,p}$ is obviously hereditary; it can be proved to be σ-additive. If $f\in L^p$, then the integral $(G_\alpha f)(x)$ exists and is finite exc. $\mathfrak{A}_{\alpha,p}$ (i.e., outside of a set $A\in\mathfrak{A}_{\alpha,p}$); we denote by $P^{\alpha,p}=P^{\alpha,p}(R^n)$, $\alpha>0$, the class of all functions $u(x)$ defined exc. $\mathfrak{A}_{\alpha,p}$ by the formula $u(x)=(G_\alpha f)(x)$ with f running over $L^p(R^n)$. For $\alpha=0$ we put $P^{0,p}=L^p$. For $u\in P^{\alpha,p}$, $u=G_\alpha f$, we define the norm $\|u\|_{\alpha,p}=\|G_\alpha f\|_{\alpha,p}=\|f\|_{L^p}$. With this norm and the exceptional class $\mathfrak{A}_{\alpha,p}$, $P^{\alpha,p}$ becomes a complete functional space (i.e., every sequence in $P^{\alpha,p}$ convergent in norm contains a subsequence convergent exc. $\mathfrak{A}_{\alpha,p}$). It can be proved that $P^{\alpha,p}$ is the perfect functional completion (i.e., the functional completion relative to the smallest exceptional class) of C_0^∞ with the norm $\|\ \|_{\alpha,p}$; on C_0^∞ the norm $\|\ \|_{\alpha,p}$ can be given by the more explicit expression $\|u\|_{\alpha,p}=\|G_{2m-\alpha}(1-\Delta)^m u\|_{L^p}$, where m is any integer such that $2m>\alpha$ and Δ is the Laplace operator.

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The following proposition gives an equivalent characterization of the class $\mathfrak{A}_{\alpha,p}$.

PROPOSITION 1. *$A\in\mathfrak{A}_{\alpha,p}$ if and only if there exists a sequence $\{u_l\}\subset C_0^\infty$, $u_l\geqq 0$, Cauchy in the norm $\|\ \|_{\alpha,p}$ and such that* $\lim_{l\to\infty} u_l(x) = +\infty$ *for all $x\in A$.*

For $\alpha>0$, k an integer, $k>\alpha$ and for $u\in L^p$, $1\leqq p<\infty$, we define

$$(\|u\|_{\alpha,p,k})^p = \|u\|_{L^p}^p + \int_{R^n} \|\Delta_t^k u\|_{L^p}^p \, |t|^{-n-p\alpha} dt, \tag{2}$$

where Δ_t^k denotes the kth forward difference with increment $t\in R^n$. It can be proved that for two integers k, k_1, $k>\alpha$, $k_1>\alpha$, the norms $\|\ \|_{\alpha,p,k}$, $\|\ \|_{\alpha,p,k_1}$ are equivalent on the subspace of L^p where they are finite. Denote this subspace by $\mathcal{B}^{\alpha,p}$. Let $0<\epsilon<\min(1,\alpha)$ and denote by $\mathfrak{B}_{\alpha,p}$ the class of all sets $A\subset R^n$ for which there exists a function $f\geqq 0$, $f\in\mathcal{B}^{\epsilon,p}$ such that $A\subset\{x\colon (G_{\alpha-\epsilon}f)(x)=+\infty\}$. $\mathfrak{B}_{\alpha,p}$ is clearly hereditary; it can be shown to be σ-additive and independent of the choice of ϵ. If $f\in\mathcal{B}^{\epsilon,p}$, then the function $u(x)=(G_{\alpha-\epsilon}f)(x)$ is defined and finite exc. $\mathfrak{B}_{\alpha,p}$. Denote by $B^{\alpha,p}$ the class of all functions u such that $u(x)=(G_{\alpha-\epsilon}f)(x)$ exc. $\mathfrak{B}_{\alpha,p}$ for some $f\in\mathcal{B}^{\epsilon,p}$. It can be proved that $B^{\alpha,p}$ with one of the equivalent norms $\|\ \|_{\alpha,p,k}$, $k>\alpha>0$, is a complete functional space rel. $\mathfrak{B}_{\alpha,p}$; moreover, it coincides with the perfect functional completion of $C_0^\infty(R^n)$ with the norm $\|\ \|_{\alpha,p,k}$. The perfect functional completion being unique, if it exists, it follows that $B^{\alpha,p}$ does not depend on ϵ occurring in the definition.

For $p=2$, both $P^{\alpha,p}$ and $B^{\alpha,p}$ coincide with the space P^α of Bessel potentials of L^2 functions.

The following proposition will be useful later.

PROPOSITION 2. *The convolution with the kernel G_β, $\beta>0$, establishes a bounded isomorphism of the space $B^{\alpha,p}$ onto $B^{\alpha+\beta,p}$. More precisely, if $u\in B^{\alpha,p}$, then $G_\beta u=v\in B^{\alpha+\beta,p}$ and every $v\in B^{\alpha+\beta,p}$ is of this form; for fixed $\beta>0$, $k>\alpha$, $k_1>\alpha+\beta$, there are two constants $c_1>0$, $c_2>0$ such that*

$$c_1\|u\|_{\alpha,p,k} \leqq \|v\|_{\alpha+\beta,p,k_1} = \|G_\beta u\|_{\alpha+\beta,p,k_1} \leqq c_2\|u\|_{\alpha,p,k}$$

for all $u\in B^{\alpha,p}$.

We also state the counterpart of Proposition 1 for the class $\mathfrak{B}_{\alpha,p}$.

PROPOSITION 3. *$A\in\mathfrak{B}_{\alpha,p}$, $\alpha>0$, $1\leqq p<\infty$, if and only if there exists a sequence $\{u_l\}\subset C_0^\infty$, $u_l\geqq 0$, Cauchy in the norm $\|\ \|_{\alpha,p,k}$, $k>\alpha$, and such that* $\lim_{l\to\infty} u_l(x)=+\infty$ *for all $x\in A$.*

Let n', n'' be two positive integers, $n=n'+n''$. For $x=(x_1, \cdots, x_n) \in R^n$ we write $x=(x_1, \cdots, x_{n'}, x_{n'+1}, \cdots, x_n)=(x', x'')$, and for a fixed $x' \in R^{n'}$ denote by $A_{x'}$ the section of a set $A \subset R^n$, $A_{x'} = \{x'' \in R^{n''}: (x', x'') \in A\}$ and by $u_{x'}$ the restriction of a function u defined on R^n, $u_{x'}(x'')=u(x', x'')$. Denote further by $\mathfrak{A}'_{\alpha,p}$, $\mathfrak{B}'_{\alpha,p}$, $\mathfrak{A}''_{\alpha,p}$, $\mathfrak{B}''_{\alpha,p}$ the exceptional classes for $P^{\alpha,p}(R^{n'})$, $B^{\alpha,p}(R^{n'})$, $P^{\alpha,p}(R^{n''})$, $B^{\alpha,p}(R^{n''})$ and by G'_α, G''_α the Bessel kernels of order α on the spaces $R^{n'}$ and $R^{n''}$, respectively.

The following theorem gives a description of sections $A_{x'}$ of sets A in $\mathfrak{A}_{\alpha,p}$ and $\mathfrak{B}_{\alpha,p}$ and of restrictions $u_{x'}$ of functions u in $P^{\alpha,p}$ and $B^{\alpha,p}$.

THEOREM. *Let* $1<p<\infty$ *and* $\alpha>0$.

(i) *If* $A\in\mathfrak{A}_{\alpha,p}$, *then* $A_{x'}\in\mathfrak{A}''_{\beta,p}$ exc. $\mathfrak{A}'_{\alpha-\beta,p}$ *for all* β, $0\leqq\beta\leqq\alpha$; *if* $u\in P^{\alpha,p}(R^n)$, *then* $u_{x'}\in P^{\beta,p}(R^{n''})$ exc. $\mathfrak{A}'_{\alpha-\beta,p}$ *for all* β, $0\leqq\beta\leqq\alpha$.

(ii) *If* $A\in\mathfrak{B}_{\alpha,p}$, *then* $A_{x'}\in\mathfrak{B}''_{\beta,p}$ exc. $\mathfrak{A}'_{\alpha-\beta,p}$ *for all* β, $0<\beta\leqq\alpha$; *also* $A_{x'}\in\mathfrak{A}''_{\beta,p}$ exc. $\mathfrak{B}'_{\alpha-\beta,p}$ *for all* β, $0\leqq\beta<\alpha$. *If* $u\in B^{\alpha,p}(R^n)$, *then* $u_{x'} \in B^{\beta,p}(R^{n''})$ exc. $\mathfrak{A}'_{\alpha-\beta,p}$ *for* $0<\beta\leqq\alpha$ *and* $u_x\in P^{\beta,p}(R^{n''})$ exc. $\mathfrak{B}'_{\alpha-\beta,p}$ *for* $0\leqq\beta<\alpha$.

Before giving the proof we make the following remark.

REMARK. In the case when $\alpha>n'/p$ it is known that $u\in B^{\alpha,p}(R^n)$ implies $u_{x'}\in B^{\alpha-n'/p}(R^{n''})$ for all $x'\in R^{n'}$. The only information we get from (ii) is that $u_{x'}\in B^{\alpha-n'/p}(R^{n''})$ exc. $\mathfrak{A}'_{n'/p,p}$, which shows that in this case the result of (ii) is not the best possible: the class $\mathfrak{A}'_{n'/p,p}$ is not empty (for $p=2$ it is the class of the sets of logarithmic capacity 0), although $\mathfrak{A}'_{\beta,p}$ is empty for all $\beta>n'/p$.

The proof of the theorem depends on the following lemmas

LEMMA 1. *For* $f\in L^p(R^n)$ *and* $0<\epsilon<1$ *let*

$$f'(x') = \left(\int_{R^{n''}} |f(x', x'')|^p dx''\right)^{1/p},$$

$$f'_\epsilon(x') = \left(\int_{R^{n''}}\int_{R^{n''}} \frac{|f(x', x''+t'') - f(x', x'')|^p}{|t''|^{n''+p\epsilon}} dx''dt''\right)^{1/p}.$$

Then (i) $f'\in L^p(R^{n'})$ *and* $\|f'\|_{L^p}=\|f\|_{L^p(R^n)}$,

(ii) $f\in\mathfrak{B}^{\epsilon,p}(R^n)$ *implies* $f'\in\mathfrak{B}^{\epsilon,p}(R^{n'})$, $f'_\epsilon\in L^p(R^{n'})$ *and* $\|f'\|_{\epsilon,p,1} \leqq C\|f\|_{\epsilon,p,1}$, $\|f'_\epsilon\|_{L^p}\leqq C\|f\|_{\epsilon,p}$, *with a constant* C *depending only on* ϵ, n', n''.

By $\|f'\|_{L^p}$, $\|f'\|_{\epsilon,p,k}$, $\|f'_\epsilon\|_{L^p}$ we understand the norms of f' as a function on $R^{n'}$.

PROOF. (i) is trivial. To prove the first part of (ii) we only need an estimate of the second term in the definition of the norm (2). We have

$$(3)\qquad \begin{aligned} &|t'|^{-\epsilon}|\Delta_{t'}f'(x')| \\ &\qquad = |t'|^{-\epsilon}\big|\, \|f(x'+t',x'')\|_{L^p(R^{n''})} - \|f(x',x'')\|_{L^p(R^{n''})}\big| \\ &\qquad \leqq \|f(x'+t',x'')-f(x',x'')\|_{L^p(R^{n''})}. \end{aligned}$$

Also, for $|t'|\neq 0$,

$$(4)\qquad \begin{aligned} \int_{R^{n''}} |t|^{-n-\epsilon p}\,dt'' &= \int_{R^{n''}} (|t'|^2+|t''|^2)^{-(n+\epsilon p)/2}\,dt'' \\ &= |t'|^{-n'-\epsilon p}\int_{R^{n''}} (1+|y''|^2)^{-(n+\epsilon p)/2}\,dy'' \\ &= C_1|t'|^{-n'-\epsilon p}. \end{aligned}$$

From (3) and (4) we get

$$\begin{aligned} &\left(\int_{R^{n'}} \|\Delta_{t'}f'\|^p_{L^p}|t'|^{-n-\epsilon p}\,dt'\right)^{1/p} \\ &\leqq \left(C_1^{-1}\int_{R^n}\int_{R^n} \frac{|f(x'+t',x'')-f(x',x'')|^p}{|t|^{n+\epsilon p}}\,dxdt\right)^{1/p} \\ &\leqq C_1^{-1/p}\Bigg[\left(\int_{R^n}\int_{R^n} \frac{|f(x'+t',x'')-f(x'+t'/2,x''+t''/2)|^p}{|t|^{n+\epsilon p}}\,dxdt\right)^{1/p} \\ &\qquad + \left(\int_{R^n}\int_{R^n} \frac{|f(x'+t'/2,x''+t''/2)-f(x',x'')|^p}{|t|^{n+\epsilon p}}\,dxdt\right)^{1/p}\Bigg] \\ &= 2^{1-\epsilon}C^{-1/p}\left(\int_{R^n}\|\Delta_t f\|^p_{L^p}t^{-\epsilon p-n}\,dt\right)^{1/p}. \end{aligned}$$

Hence, $\|f'\|_{\epsilon,p,1}\leqq \max(1,\,2^{1-\epsilon}C_1^{-1})\|f\|_{\epsilon,p,1}$.

The proof of the second part of (ii) is obtained by interchanging the roles of x' and x''.

As usual S denotes the class of all C^∞ functions of rapid decrease.

LEMMA 2. *Let $\gamma\geqq 0$, $\delta\geqq 0$, $1<p<\infty$ be fixed. The equation*

$$(5)\qquad \begin{aligned} &\int_{R^n} G_{\gamma+\delta}(x-y)g(y)\,dy \\ &\qquad = \int_{R^{n'}}\int_{R^{n''}} G'_\gamma(x'-y')G''_\delta(x''-y'')f(y',y'')\,dy'dy'' \end{aligned}$$

establishes a 1-1 *mapping* $f \to g$ *of* S *onto itself. There exists a constant* C *such that* (i) $\|f\|_{L^p} \leqq C\|g\|_{L^p}$,

(ii) $\|f\|_{\eta,p,k} \leqq C\|g\|_{\eta,p,k}$, *for all* $g \in S$, $0 < \eta < k$.

PROOF. Rewriting (5) in terms of Fourier transforms and using (1) we get

$$\hat{f}(\xi) = \frac{(1+|\xi'|^2)^{\gamma/2}(1+|\xi''|^2)^{\delta/2}}{(1+|\xi|^2)^{(\gamma+\delta)/2}}\hat{g}(\xi). \tag{6}$$

The first statement of the lemma is now trivial. Using the Mihlin theorem [3] about multipliers of Fourier transforms we verify immediately that the coefficient of $\hat{g}(\xi)$ on the right hand side of (6) is a multiplier of type (p, p) for every p, $1 < p < \infty$, which proves (i). (ii) is obtained from (i) by replacing f and g by $\Delta_t^k f$ and $\Delta_t^k g$, respectively, with an arbitrary fixed $t \in R^n$.

PROOF OF THE THEOREM. Let $\gamma \geqq 0$, $\delta \geqq 0$ and $u, f, g \in S$ be related by the equation

$$\begin{aligned} u(x', x'') &= \int_{R^n} G_{\gamma+\delta}(x-y)g(y)\,dy \\ &= \int_{R^{n'}}\int_{R^{n''}} G'_\gamma(x'-y')G''_\delta(x''-y'')f(y', y'')\,dy'dy'' \\ &= \int_{R^{n''}} G''_\delta(x''-y'')\left[\int_R G'_\gamma(x'-y')f(y', y'')dy'\right]dy''. \end{aligned} \tag{7}$$

Let $\alpha > 0$ and $0 \leqq \beta \leqq \alpha$; we put, in (7), $\delta = \beta$, $\gamma = \alpha - \beta$. By the definition of the norm $\| \ \|_{\beta,p}$, we have, for every $x' \in R^{n'}$,

$$\|u_{x'}\|_{\beta,p} = \left\|\int_{R^{n'}} G'_{\alpha-\beta}(x'-y')f(y', y'')\,dy'\right\|_{L^p}$$

and using the continuous version of the Minkowski inequality

$$\begin{aligned} \|u_{x'}\|_{\beta,p} &\leqq \int_{R^{n'}} G'_{\alpha-\beta}(x'-y')\|f(y', y'')\|_{L^p(R^{n''})}\,dy' \\ &= \int_{R^{n'}} G'_{\alpha-\beta}(x'-y')f'(y')\,dy'. \end{aligned} \tag{8}$$

Let now $\alpha > 0$, $0 < \beta \leqq \alpha$ and $\epsilon = (1/2)\min(1, \beta)$. Choose, in (7), $\delta = \beta - \epsilon$, $\gamma = \alpha - \beta$. Using Proposition 2, we get, with an integer $k > \beta$ and a constant C depending only on β, and n'' k,

$$\|u_{x'}\|_{\beta,p,k} \leqq C\left\|\int_{R^n} G'_{\alpha-\beta}(x'-y')f(y', y'')\,dy'\right\|_{\epsilon,p,1},$$

and, using again a continuous form of the Minkowski inequality,

$$\|u_{x'}\|_{\beta,p,k} \leqq C\left[\left(\int_{R^{n'}} G'_{\alpha-\beta}(x'-y')f'(y')\,dy'\right)^p + \left(\int_{R^{n'}} G'_{\alpha-\beta}(x'-y')f'_\epsilon(y')\,dy'\right)^p\right]^{1/p} \leqq C\left[\int_{R^{n'}} G'_{\alpha-\beta}(x'-y')[f'(y')+f'_\epsilon(y')]\,dy'\right]. \tag{9}$$

Finally, let $\alpha>0$, $0\leqq\beta<\alpha$ and $\epsilon=(1/2)\min(1,\alpha-\beta)$. Put, in (7), $\delta=\beta$, $\gamma=\alpha-\beta-\epsilon$. We get, similarly, as in (8),

$$\|u_{x'}\|_{\beta,p} \leqq \int_{R^{n'}} G'_{\alpha-\beta-\epsilon}(x'-y')f'(y')\,dy'. \tag{10}$$

In the inequalities (8), (9), and (10), f' and f'_ϵ are defined as in Lemma 1.

We shall now prove (ii). Let $A\in\mathfrak{B}_{\alpha,p}$. By Proposition 3, there is a sequence $\{u_l\}\subset C_0^\infty$, $u_l\geqq 0$, Cauchy in $B^{\alpha,p}$ and such that $\lim_{l\to\infty} u_l(x) =+\infty$ for all $x\in A$. Let $\epsilon=(1/2)\min(1,\beta)$ and $\{f_l\}$, $\{g_l\}\subset S$ be such that

$$u_l(x) = \int_{R^n} G_{\alpha-\epsilon}(x-y)g_l(y)\,dy = \int_{R^{n''}} G''_{\beta-\epsilon}(x''-y'')\int_{R^{n'}} G'_{\alpha-\beta}(x'-y')f_l(y',y'')\,dy'dy''. \tag{11}$$

By Proposition 2, we may assume, without loss of generality, that $\sum_{l=1}^\infty \|g_{l+1}-g_l\|_{\epsilon,p,1}<\infty$, consequently, by Lemmas 1 and 2, $h' =\sum_{l=1}^\infty (f_{l+1}-f_l)'+\sum_{l=1}^\infty (f_{l+1}-f_l)'_\epsilon\in L^p(R^{n'})$. We conclude, using (9), that the sequence $(u_l)_{x'}$ is Cauchy in $B^{\beta,p}(R^{n''})$ for every x' outside of the set $A'=\{x'\in R^{n'}: \int_{R^{n'}}G'_{\alpha-\beta}(x'-y')h'(y')\,dy'=+\infty\}\in\mathfrak{A}'_{\alpha-\beta,p}$. This proves, using the functional space property, that $A_{x'}\in\mathfrak{B}''_{\beta,p}$ for $x'\notin A'$ i.e., exc. $\mathfrak{A}'_{\alpha-\beta,p}$.

Let now $u\in B^{\beta,p}$ and choose an integer $k>\alpha$. By the functional space property there is a sequence $\{u_l\}\subset C_0^\infty$ such that $\lim_{l\to\infty}\|u-u_l\|_{\alpha,p,k}=0$ and $\lim_{l\to\infty} u_l(x)=u(x)$ exc. $\mathfrak{B}_{\alpha,p}$. Let $A\in\mathfrak{B}_{\alpha,p}$ be the union of the exceptional set of u (i.e., the set where u is not defined or infinite) and the set where $\{u_l(x)\}$ does not converge to $u(x)$. Let $A'\in\mathfrak{A}'_{\alpha-\beta,p}$ be the set with the property that $A_{x'}\in\mathfrak{B}''_{\beta,p}$ for $x'\notin A'$. Assume without loss of generality that $\sum_{l=1}^\infty\|u_{l+1}-u_l\|_{\alpha,p,k}<\infty$, define $\{f_l\}$, $\{g_l\}\subset S$ as in (11) and let

$$h' = \sum_{l=1}^{\infty} (f_{l+1} - f_l)' + \sum_{l=1}^{\infty} (f_{l+1} - f_l)'_\epsilon \in L^p(R^{n'}).$$

It follows from (9) that $(u_l)_{x'}$ is Cauchy in $B^{\beta,p}(R^{n''})$ for every x' outside of the set $B' = \{x' \in R^{n'}: \int G'_{\alpha-\beta}(x'-y')h'(y')\,dy' = +\infty\}$. Consequently, for $x' \notin A' \cup B'$, $(u_l)_{x'}$ is Cauchy in $B^{\beta,p}(R^{n''})$ and converges to $u_{x'}$ exc. $\mathfrak{B}''_{\beta,p}$. Since $A' \cup B' \in \mathfrak{A}'_{\alpha-\beta,p}$ this proves that $u_{x'} \in B^{\beta,p}(R^{n''})$ exc. $\mathfrak{A}'_{\alpha-\beta,p}$. The proofs of the remaining part of (ii) and of (i) follow the same idea and are even simpler. We use the statements (i) of Lemmas 1 and 2 and inequality (8) to prove (i) and the statements (ii) of Lemmas 1, 2 and inequality (10) to prove the remainder of (ii).

References

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