NOTE ON THE UNITS OF A REAL QUADRATIC FIELD

S. CHOWLA

Let $p$ denote a prime $\equiv 1 \pmod{4}$. It is known that, if $\theta = e^{2\pi i/p}$,
\[
\prod_{n \in b} (1 - \theta^n) \prod_{r \in a} (1 - \theta^r) = \eta
\]
(here $b$ and $a$ run over the quadratic nonresidues and quadratic residues, respectively, that lie between 0 and $p$) is a unit of the real quadratic field $\mathbb{R}(\sqrt{p})$, and that $\eta > 1$.

The fact that $\eta > 1$ is usually deduced from the theory of the class-number of quadratic fields. We present a short proof independent of the theory of the class-number. As in the paper of Chowla and Mordell [Note on the nonvanishing of $L(1)$, Proc. Amer. Math. Soc. 12 (1961), 283–284], we have $S(1) \neq 0$ (proved in the paper cited) in the relation [ibid.]

\[(1) \quad \eta = \exp\{S L(1)\},\]

where $L(1) = \sum_{\mathbb{Z}^*} (n \mid p) n^{-1}$ and $S = \sum_{\mathbb{Z}^*} (n \mid p) \theta^n$. Here $(n \mid p)$ is Legendre's symbol.

From the theory of the Gaussian sum,

\[(2) \quad S = \sqrt{p}.
\]

Also,

\[
L(1) = \lim_{s \to 1+0} L(s) \geq 0
\]

since

\[
L(s) = \sum_{1}^{\infty} (n \mid p) n^{-s} > 0 \quad (s > 1)
\]

from the Euler product for $L(s)$. It now follows from Chowla and Mordell that

\[(3) \quad L(1) > 0.
\]

From (1), (2), (3) we obtain $\eta > 1$. Q.E.D.

Pennsylvania State University

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