NOTE ON THE UNITS OF A REAL QUADRATIC FIELD

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Let p denote a prime $\equiv 1 \pmod 4$. It is known that, if $\theta = e^{2\pi i/p}$,

$$\frac{\prod_{n} (1 - \theta^b)}{\prod_{r} (1 - \theta^a)} = \eta$$

(here b and a run over the quadratic nonresidues and quadratic residues, respectively, that lie between 0 and p) is a unit of the real quadratic field $R(\sqrt{p})$, and that $\eta > 1$.

The fact that $\eta > 1$ is usually deduced from the theory of the class-number of quadratic fields. We present a short proof independent of the theory of the class-number. As in the paper of Chowla and Mordell [*Note on the nonvanishing of* $L(1)$, Proc. Amer. Math. Soc. **12** (1961), 283–284], we have $\eta \neq 1$ since $L(1) \neq 0$ (proved in the paper cited) in the relation [ibid.]

$$\eta = \exp\{SL(1)\}, \tag{1}$$

where $L(1) = \sum_1^\infty (n|p)n^{-1}$ and $S = \sum_1^p (n|p)\theta^n$. Here $(n|p)$ is Legendre's symbol.

From the theory of the Gaussian sum,

$$S = \sqrt{p}. \tag{2}$$

Also,

$$L(1) = \lim_{s \to 1+0} L(s) \geq 0$$

since

$$L(s) = \sum_1^\infty (n \mid p)n^{-s} > 0 \qquad (s > 1)$$

from the Euler product for $L(s)$. It now follows from Chowla and Mordell that

$$L(1) > 0. \tag{3}$$

From (1), (2), (3) we obtain $\eta > 1$. Q.E.D.

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