A NOTE ON THE ERGODIC THEOREM

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Let Ω be a set, $\mathfrak{F}$ a σ-algebra of subsets of Ω, and T a 1-1 transformation which maps Ω onto Ω and is bimeasurable, i.e., $A \in \mathfrak{F}$ if and only if $TA \in \mathfrak{F}$. Let P be a probability measure defined on $\mathfrak{F}$. P is said to be invariant provided $P(A) = P(TA)$ for all $A \in \mathfrak{F}$. For such probability measures the individual ergodic theorem holds, i.e., if f is a measurable function defined on Ω such that $\int_\Omega |f| \, dP < \infty$ then $\lim_{n\to\infty} (1/n) \sum_{i=0}^{n-1} f(T^i x)$ exists for almost all x. Let $\mathfrak{B}$ be the class of bounded measurable functions defined on Ω. In this note we show that the individual ergodic theorem holds for members of $\mathfrak{B}$ for a class of probability measures which includes the invariant ones and is considerably wider than that class.

Let $M = (m_{i,j}, \; i, j = 0, 1, 2, \cdots)$ be an infinite matrix satisfying:

(i) $m_{i,j} \geqq 0$ for all i and j,

(ii) $\sum_{j=0}^{\infty} m_{i,j} \leqq C < \infty$ uniformly in i,

(iii) $\lim_{i\to\infty} m_{i,j} = 0$ for all j,

(iv) $\lim_{i\to\infty} \sum_{j=0}^{\infty} m_{i,j} = 1$.

Such a matrix M represents a positive regular summation method, and a good deal is known about such methods. In particular, such methods preserve positivity and convergence of sequences. See, e.g., Hardy [**4**]. We shall call such a method shift-invariant provided for every sequence $\{a_n\}$ of real numbers for which $\lim_{n\to\infty} \sum_{j=0}^{\infty} m_{n,j} a_j = a$ exists and is finite we have $\lim_{n\to\infty} \sum_{j=0}^{\infty} m_{n,j} a_{j+1} = a$. The (C, 1) method, i.e. $m_{i,j} = 1/(i+1)$ for $j \leqq i$ and $m_{i,j} = 0$ for $j > i$, is an example of a shift-invariant method. We shall say the sequence $\{a_n\}$ is M-summable if $\lim_{n\to\infty} \sum_{j=0}^{\infty} m_{n,j} a_j$ exists.

THEOREM. *Let P be a probability measure defined on $\mathfrak{F}$ and let M be a positive, regular, shift-invariant summation method. If the sequence $\{P(T^n A)\}$ is M-summable for each $A \in \mathfrak{F}$ then the ergodic theorem holds for each $f \in \mathfrak{B}$.*

PROOF. For each n and each $A \in \mathfrak{F}$ let $Q_n(A) = \sum_{j=0}^{\infty} m_{n,j} P(T^j A)$ and let $Q(A) = \lim_{n\to\infty} Q_n(A)$. Then it follows from known results (see, e.g., Halmos [**3**, p. 170, Problem 14]) that $Q(A)$ is a probability measure. Also since M is shift-invariant we see that Q is invariant. Now let $A \in \mathfrak{F}$ be such that $Q(A) = 0$ and let $B = \bigcup_{i=-\infty}^{\infty} T^i A$. Then

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$Q(B)=0$ and B is such that $T^iB=B$ for all integers i. Hence $P(T^nB)=P(B)$ for all n and, consequently, $P(B)=Q(B)=0$. It follows that P is absolutely continuous with respect to Q. Now if $f\in\mathfrak{B}$ then $\int_\Omega|f|\,dQ<\infty$ and the ergodic theorem holds for f with respect to Q. But from the absolute continuity of P with respect to Q we see that the ergodic theorem also holds for f with respect to P. The theorem is proved.

Note that if the ergodic theorem holds for each $f\in\mathfrak{B}$ then it follows from the Lebesgue bounded convergence theorem that the sequence $\{P(T^nA)\}$ is (C, 1)-summable for each $A\in\mathfrak{F}$. Thus we have

COROLLARY 1. *A necessary and sufficient condition that the ergodic theorem hold for each $f\in\mathfrak{B}$ is that the sequence $\{P(T^nA)\}$ be* (C, 1)-*summable for each $A\in\mathfrak{F}$.*

Another immediate consequence of the theorem is

COROLLARY 2. *Let M be a positive, regular, shift-invariant summation method. If the sequence $\{P(T^nA)\}$ is M-summable for each $A\in\mathfrak{F}$ then it is also* (C, 1)-*summable.*

Corollary 1 is not new. See, e.g., Dowker [2], and Brunk [1].

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