ON MEASURE AND OTHER PROPERTIES OF A HAMEL BASIS

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F. B. Jones has established [2] several interesting measure-related properties of a Hamel basis. In particular, he established the existence of a Hamel basis which contains a nonempty perfect set (and, hence, supports a nontrivial Borel measure) and commented on the plausibility of the notion that a Hamel basis should be in some sense "thick." Our purpose is to complement Jones' results by showing, subject to the continuum hypothesis which we assume throughout, that there exists a Hamel basis which intersects each first category set in, at most, a countable set and, hence, has universal measure zero (cf. [1]). It then follows, for instance, that the sum \( E + F = \{ e + f; e \in E, f \in F \} \) of a universal null set \( E \) and a universal null set \( F \) need not be a universal null set and, moreover, an iterate \( E^* \) of \( E \) need not be Lebesgue measurable (cf. [2]).

In order to fulfill our purpose it suffices to establish the following theorem. (We wish to acknowledge collaboration with R. E. Zink on problems related to the content of this note.)

**Theorem.** There exists a Hamel basis \( H \) which intersects each perfect nowhere dense set in, at most, a countable set.

Before proceeding to a proof of the theorem we wish to state the following lemma which we shall have occasion to use.

**Lemma.** If \( Q \) is a first category subset of \((0, 1]\) and \( x \) is a point of \((0, 1)\), then there exists a point \( y \) of \((0, x)\) such that \( x + y \in (x, 1) \) and neither \( y \) nor \( x + y \) is an element of \( Q \).

**Proof of Theorem.** Let \( \Omega \) denote the first uncountable ordinal and let \( \{ P_\alpha \}_{\alpha \in \Omega} \) and \( \{ x_\alpha \}_{\alpha \in \Omega}, x_1 = 1, \) be well orderings of the perfect nowhere dense subsets of \((0, 1]\) and the points of \((0, 1]\). We shall define \( H = \bigcup_{\alpha \in \Omega} H_\alpha \) inductively as follows. Let \( H_1 = \{ 1 \}, R_1 = \emptyset \). Suppose \( 1 < \alpha < \Omega \) and, for \( 1 \leq \beta < \alpha \), \( H_\beta \) and \( R_\beta \) satisfy:

1. \( H_\beta \) is, at most, countable.
2. The elements of \( H_\beta \) are linearly independent over the rationals.
3. \( R_\beta \) is a subset of the linear span \( H_\beta^L \) of \( H_\beta \).
4. \( H_\beta \cap R_\beta = \emptyset \).

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(5) \((H_\beta \cup R_\beta) \cup \{x_\gamma\}\).
(6) \(R_\delta \subseteq \{x_\gamma\}\).
(7) \(H_\gamma \subseteq H_\beta, R_\gamma \subseteq R_\beta, \gamma < \beta\).
(8) If \(x \in H_\beta - H_\gamma\), then \(x \in P_\gamma (\beta > \gamma)\).

In order to simplify what follows, let \(K_a = \bigcup_{\beta < a} H_\beta\), \(S_a = \bigcup_{\beta < a} R_\beta\), and \(T_a = \bigcup_{\beta < a} P_\beta\). We now consider cases:

(a) If \(x_\alpha \in K_a\), let \(H_a = K_a\) and \(R_a = S_a\).
(b) If \(x_\alpha \in K^L_a - K_a\), let \(H_a = K_a\) and \(R_a = S_a \cup \{x_\alpha\}\).
(c) If \(x_\alpha \in K^L_a \cup T_a\), let \(H_a = K_a \cup \{x_\alpha\}\) and \(R_a = S_a\).
(d) If \(x_\alpha \in T_a - K^L_a\) and there exists a rational number \(r\) such that \(rx_\alpha \in (0, 1) - T_a\), let \(\lambda\) be the least index such that \(x_\lambda\) is a rational multiple of \(x_\alpha\) in \((0, 1) - T_a\) and then let \(H_a = K_a \cup \{x_\lambda\}\) and \(R_a = S_a \cup \{x_\alpha\}\).
(e) If \(\{x_\alpha\} \cap (0, 1) \subseteq T_a - K^L_a\), let \(Q = (K_a \cup \{x_\alpha\}) \cup T_a\) and apply the lemma to obtain the least index \(\lambda\) such that \(x_\alpha\) and \(x_\lambda\) play the role of \(x\) and \(y\) of the lemma. Then let \(H_a = K_a \cup \{x_\lambda\} \cup \{x_\lambda + x_\alpha\}\) and \(R_a = T_a \cup \{x_\alpha\}\).

**Bibliography**


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