F. B. Jones has established [2] several interesting measure-related properties of a Hamel basis. In particular, he established the existence of a Hamel basis which contains a nonempty perfect set (and, hence, supports a nontrivial Borel measure) and commented on the plausibility of the notion that a Hamel basis should be in some sense "thick." Our purpose is to complement Jones' results by showing, subject to the continuum hypothesis which we assume throughout, that there exists a Hamel basis which intersects each first category set in, at most, a countable set and, hence, has universal measure zero (cf. [1]). It then follows, for instance, that the sum \( E + F = \{ e + f ; e \in E, f \in F \} \) of a universal null set \( E \) and a universal null set \( F \) need not be a universal null set and, moreover, an iterate \( E^k \) of \( E \) need not be Lebesgue measurable (cf. [2]).

In order to fulfill our purpose it suffices to establish the following theorem. (We wish to acknowledge collaboration with R. E. Zink on problems related to the content of this note.)

**Theorem.** There exists a Hamel basis \( H \) which intersects each perfect nowhere dense set in, at most, a countable set.

Before proceeding to a proof of the theorem we wish to state the following lemma which we shall have occasion to use.

**Lemma.** If \( Q \) is a first category subset of \((0, 1)\) and \( x \) is a point of \((0, 1)\), then there exists a point \( y \) of \((0, x)\) such that \( x+y \in (x, 1) \) and neither \( y \) nor \( x+y \) is an element of \( Q \).

**Proof of Theorem.** Let \( \Omega \) denote the first uncountable ordinal and let \( \{ P_\alpha \}_{\alpha < \Omega} \) and \( \{ x_\alpha \}_{\alpha < \Omega}, x_1 = 1, \) be well orderings of the perfect nowhere dense subsets of \((0, 1)\) and the points of \((0, 1)\). We shall define \( H = \bigcup_{\alpha < \Omega} H_\alpha \) inductively as follows. Let \( H_1 = \{ 1 \}, R_1 = \emptyset \). Suppose \( 1 < \alpha < \Omega \) and, for \( 1 \leq \beta < \alpha, H_\beta \) and \( R_\beta \) satisfy:

1. \( H_\beta \) is, at most, countable.
2. The elements of \( H_\beta \) are linearly independent over the rationals.
3. \( R_\beta \) is a subset of the linear span \( H_\beta^0 \) of \( H_\beta \).
4. \( H_\beta \cap R_\beta = \emptyset \).

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(5) \((H_\beta \cup R_\beta) \supset \{x_\gamma \}\).
(6) \(R_\beta \subset \{x_\gamma \}\).
(7) \(H_\gamma \subset H_\beta, R_\gamma \subset R_\beta, \gamma < \beta\).
(8) If \(x \in H_\beta - H_\gamma\), then \(x \notin P_\gamma (\beta > \gamma)\).

In order to simplify what follows, let \(K_\alpha = \bigcup_{\beta < \alpha} H_\beta\), \(S_\alpha = \bigcup_{\beta < \alpha} R_\beta\), and \(T_\alpha = \bigcup_{\beta < \alpha} P_\beta\). We now consider cases:

(a) If \(x_\alpha \in K_\alpha\), let \(H_\alpha = K_\alpha\) and \(R_\alpha = S_\alpha\).

(b) If \(x_\alpha \in K_\alpha^L - K_\alpha\), let \(H_\alpha = K_\alpha\) and \(R_\alpha = S_\alpha \cup \{x_\alpha\}\).

(c) If \(x_\alpha \in K_\alpha^L \cup T_\alpha\), let \(H_\alpha = K_\alpha \cup \{x_\alpha\}\) and \(R_\alpha = S_\alpha\).

(d) If \(x_\alpha \in T_\alpha - K_\alpha^L\) and there exists a rational number \(r\) such that \(rx_\alpha \in (0, 1) - T_\alpha\), let \(\lambda\) be the least index such that \(x_\lambda\) is a rational multiple of \(x_\alpha\) in \((0, 1) - T_\alpha\) and then let \(H_\alpha = K_\alpha \cup \{x_\lambda\}\) and \(R_\alpha = S_\alpha \cup \{x_\alpha\}\).

(e) If \([x_\alpha]^L \cap (0, 1) \subset T_\alpha - K_\alpha^L\), let \(Q = (K_\alpha \cup \{x_\alpha\})^L \cup T_\alpha\) and apply the lemma to obtain the least index \(\lambda\) such that \(x_\lambda\) and \(x_\alpha\) play the role of \(x\) and \(y\) of the lemma. Then let \(H_\alpha = K_\alpha \cup \{x_\lambda\} \cup \{x_\lambda + x_\alpha\}\) and \(R_\alpha = T_\alpha \cup \{x_\alpha\}\).

**Bibliography**


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