SOME CLASSES OF REGULAR UNIVALENT FUNCTIONS

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1. **Introduction.** Let $\mathcal{S}$ denote the family of functions f which are regular and univalent in the unit disk $|z|<1$, hereafter called Δ, and which satisfy the conditions $f(0)=0$ and $f'(0)=1$, and let $\mathcal{S}^*$, $\mathcal{K}$ and $\mathcal{C}$ be the subfamilies of $\mathcal{S}$ whose members map Δ onto domains which are starlike with respect to the origin, convex, and close-to-convex, respectively. Then, as was shown by W. Kaplan [2],

$$\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{C} \subset \mathcal{S}. \tag{1.1}$$

Recently, in a seminar given at Rutgers University, Professor M. S. Robertson showed that the starlike function k, where $k(z)=z(1-z)^{-2}$, has the property that $2z^{-1}\int_0^z k(t)\,dt$, $z\in\Delta$, defines a function in $\mathcal{S}^*$. The extremal character of the Koebe function, k, within the class $\mathcal{S}^*$, suggests the following generalization.

THEOREM 1. *If s is in $\mathcal{S}^*$, then the function S, defined by $S(z)=(2/z)\int_0^z s(t)\,dt$, is likewise in $\mathcal{S}^*$.*

It is the purpose of this note to establish Theorem 1 and to consider similar conclusions for other members of $\mathcal{S}$.

2. **Preliminary results.** The class of all regular functions P which satisfy the conditions $P(0)=1$ and $\operatorname{Re}\{P(z)\}>0$, for z in Δ, is represented by $\mathcal{P}$.

LEMMA 1. *If N and D are regular in Δ, $N(0)=D(0)=0$, D maps Δ onto a many-sheeted region which is starlike with respect to the origin, and $N'/D'\in\mathcal{P}$, then $N/D\in\mathcal{P}$.*

REMARK. The essential ideas in the proof of Lemma 1 are the same as given by Sakaguchi in the case D is univalent [6]. R. M. Robinson [5, Lemma, p. 30] has used a similar technique.

PROOF. By known properties of class $\mathcal{P}$, [4], we can write

$$\left|\frac{N'(z)}{D'(z)}-a(r)\right|<a(r),\quad |z|<r,\qquad 0<r<1.$$

Choose $A(z)$ so that

$$D'(z)A(z)=N'(z)-a(r)D'(z)\quad\text{and}\quad |A(z)|<a(r),$$

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for $|z|<r$. Fix z_0, $z_0 \in \Delta$, and let L be the segment joining 0 to $D(z_0)$ which lies in one sheet of the starlike image of Δ by the mapping D. Let L^{-1} be the pre-image of L under D and let $r=\max|z|$, where $z \in L^{-1}$. Then

$$\begin{aligned} |N(z_0)-a(r)D(z_0)| &= \left|\int_0^{z_0}[N'(t)-a(r)D'(t)]\,dt\right| \\ &= \left|\int_{L^{-1}} D'(t)A(t)\,dt\right| \leqq a(r)\int_L |dD(t)| \\ &= a(r)\,|D(z_0)|\,. \end{aligned}$$

This proves the lemma.

LEMMA 2. *If $s \in \mathcal{S}^*$, then $\sigma(z)=\int_0^z s(t)\,dt$, $z \in \Delta$, gives a function which is 2-valently starlike with respect to the origin for all z in Δ.*

PROOF. Let $D(z)=z\sigma'(z)=zs(z)$ and $N(z)=\sigma(z)$, then D is (2-valently) starlike with respect to the origin since

$$\operatorname{Re}\left\{\frac{zD'(z)}{D(z)}\right\} = \operatorname{Re}\left\{1+\frac{zs'(z)}{s(z)}\right\} > 1 > 0 \quad \text{for} \quad z \text{ in } \Delta.$$

Furthermore,

$$\operatorname{Re}\left\{\frac{N'(z)}{D'(z)}\right\} > 0, \qquad z \in \Delta,$$

because

$$\operatorname{Re}\left\{\frac{D'(z)}{N'(z)}\right\} = \operatorname{Re}\left\{1+\frac{zs'(z)}{s(z)}\right\} > 0.$$

An application of Lemma 1 [which is valid even though $N'(0)/D'(0) \neq 1$] yields

$$\operatorname{Re}\left\{\frac{N(z)}{D(z)}\right\} > 0, \quad \text{or} \quad \operatorname{Re}\left\{\frac{z\sigma'(z)}{\sigma(z)}\right\} > 0, \quad \text{for} \quad z \text{ in } \Delta;$$

and this together with

$$\int_0^{2\pi} \operatorname{Re}\left\{\frac{re^{i\theta}\sigma'(re^{i\theta})}{\sigma(re^{i\theta})}\right\} d\theta = 2\pi\left\{\frac{z\sigma'(z)}{\sigma(z)}\right\}_{z=0} = 4\pi, \qquad 0 < r < 1,$$

which follows from the mean-value theorem for harmonic functions, shows that σ is 2-valent and starlike [1, p. 212]. One can show, furthermore, that σ is convex [1] throughout the unit disk.

3. Theorems and proofs.

PROOF OF THEOREM 1. For the function S, defined in Theorem 1, we have

$$\frac{zS'(z)}{S(z)} = \frac{zs(z) - \int_0^z s(t)\,dt}{\int_0^z s(t)\,dt} = \frac{z\sigma'(z) - \sigma(z)}{\sigma(z)}$$

and then differentiation of the numerator and the denominator of the last expression gives

$$\frac{[z\sigma'(z) - \sigma(z)]'}{\sigma'(z)} = \frac{z\sigma''(z)}{\sigma'(z)} = \frac{zs'(z)}{s(z)}.$$

An application of Lemma 1 and Lemma 2 completes the proof.

As an immediate consequence of Theorem 1 and the fact that

$$f \in \mathcal{K} \quad \text{if and only if} \quad zf' \in \mathcal{S}^*, \tag{3.1}$$

we have the following corollary.

COROLLARY 1.1. *If* $s \in \mathcal{S}^*$, *then* $\int_0^z (2/t^2)\left[\int_0^t s(x)dx\right] dt$ *defines a member of* $\mathcal{K}$.

THEOREM 2. *If* $c \in \mathcal{K}$ *and* $C(z) = (2/z)\int_0^z c(t)\,dt$, *then* $C \in \mathcal{K}$.

PROOF. Let $s(z) = zc'(z)$; then [by (3.1)] $s \in \mathcal{S}^*$. Let S be defined as in Theorem 1, then

$$\begin{aligned} S(z) &= \frac{2}{z}\int_0^z tc'(t)\,dt = \frac{2}{z}\left[zc(z) - \int_0^z c(t)\,dt\right] \\ &= z\left[\frac{2c(z)}{z} - \frac{2}{z^2}\int_0^z c(t)\,dt\right] = zC'(z). \end{aligned}$$

Since S is starlike, it follows from (3.1) that C is convex. Theorem 2 can be proved directly from Lemma 1 by the method of Theorem 1.

COROLLARY 2.1. *If* $c \in \mathcal{K}$, $C(z) = (2/z)\int_0^z c(t)\,dt$ *and* $h(z) = 2c(z) - C(z)$, *then* $h \in \mathcal{S}^*$.

f is in $\mathcal{C}$ if, and only if, there exists a function g such that $g(z) = \epsilon s(z)$, $s \in \mathcal{S}^*$, $|\epsilon| = 1$, and

$$\operatorname{Re}\left\{\frac{zf'(z)}{g(z)}\right\} > 0, \qquad \text{for } z \text{ in } \Delta.$$

In this case, we say f is close-to-convex with respect to the (starlike) function g.

THEOREM 3. *If f is close-to-convex with respect to g,*

$$F(z) = \frac{2}{z}\int_0^z f(t)\,dt \quad \textit{and} \quad G(z) = \frac{2}{z}\int_0^z g(t)\,dt,$$

then F is close-to-convex with respect to G.

The proof is similar to that of Theorem 1.

In [3], J. Krzyż showed that the radius of close-to-convexity of every function in $\mathcal{S}$ is greater than or equal to r_0, $.80 < r_0 < .81$.

THEOREM 4. *If $f \in \mathcal{S}$ and $F(z) = (2/z)\int_0^z f(t)\,dt$, then F is schlicht (and close-to-convex) for $|z| < r$, $r \geqq r_0$.*

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