SOME CLASSES OF REGULAR UNIVALENT FUNCTIONS

R. J. LIBERA

1. Introduction. Let $S$ denote the family of functions $f$ which are regular and univalent in the unit disk $|z| < 1$, hereafter called $\Delta$, and which satisfy the conditions $f(0) = 0$ and $f'(0) = 1$, and let $S^*$, $\mathcal{K}$ and $\mathcal{C}$ be the subfamilies of $S$ whose members map $\Delta$ onto domains which are starlike with respect to the origin, convex, and close-to-convex, respectively. Then, as was shown by W. Kaplan [2],

\begin{equation}
\mathcal{K} \subset S^* \subset \mathcal{C} \subset S.
\end{equation}

Recently, in a seminar given at Rutgers University, Professor M. S. Robertson showed that the starlike function $k$, where $k(z) = z(1 - z)^{-2}$, has the property that $2z^{-1}\int_0^z k(t) \, dt, z \in \Delta$, defines a function in $S^*$. The extremal character of the Koebe function, $k$, within the class $S^*$, suggests the following generalization.

**Theorem 1.** If $s$ is in $S^*$, then the function $S$, defined by $S(z) = (2/z)\int_0^z s(t) \, dt$, is likewise in $S^*$.

It is the purpose of this note to establish Theorem 1 and to consider similar conclusions for other members of $S$.

2. Preliminary results. The class of all regular functions $P$ which satisfy the conditions $P(0) = 1$ and $\Re\{P(z)\} > 0$, for $z$ in $\Delta$, is represented by $\mathcal{P}$.

**Lemma 1.** If $N$ and $D$ are regular in $\Delta$, $N(0) = P(0) = 0$, $D$ maps $\Delta$ onto a many-sheeted region which is starlike with respect to the origin, and $N'/D' \in \mathcal{P}$, then $N/D \in \mathcal{P}$.

**Remark.** The essential ideas in the proof of Lemma 1 are the same as given by Sakaguchi in the case $D$ is univalent [6]. R. M. Robinson [5, Lemma, p. 30] has used a similar technique.

**Proof.** By known properties of class $\mathcal{P}$, [4], we can write

\[ \left| \frac{N'(z)}{D'(z)} - a(r) \right| < a(r), \quad |z| < r, \quad 0 < r < 1. \]

Choose $A(z)$ so that

\[ D'(z)A(z) = N'(z) - a(r)D'(z) \quad \text{and} \quad |A(z)| < a(r), \]

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for $|z| < r$. Fix $z_0, z_0 \in \Delta$, and let $L$ be the segment joining 0 to $D(z_0)$ which lies in one sheet of the starlike image of $\Delta$ by the mapping $D$. Let $L^{-1}$ be the pre-image of $L$ under $D$ and let $r = \max |z|$, where $z \in L^{-1}$. Then

$$
|N(z_0) - a(r)D(z_0)| = \left| \int_0^{z_0} [N'(t) - a(r)D'(t)] \, dt \right|
= \left| \int_{L^{-1}} D'(t)A(t) \, dt \right| \leq a(r) \int_{L} |dD(t)|
= a(r) \left| D(z_0) \right|.
$$

This proves the lemma.

**Lemma 2.** If $s \in \mathbb{S}^*$, then $\sigma(z) = \int_0^s t \, dt, z \in \Delta$, gives a function which is 2-valently starlike with respect to the origin for all $z$ in $\Delta$.

**Proof.** Let $D(z) = zs'(z) = zs(z)$ and $N(z) = a(z)$, then $D$ is (2-valently) starlike with respect to the origin since

$$
Re\left\{ \frac{sD'(z)}{D(z)} \right\} = Re\left\{ 1 + \frac{zs'(z)}{s(z)} \right\} > 1 > 0 \quad \text{for} \quad z \in \Delta.
$$

Furthermore,

$$
Re\left\{ \frac{N'(z)}{D'(z)} \right\} > 0, \quad z \in \Delta,
$$

because

$$
Re\left\{ \frac{D'(z)}{N'(z)} \right\} = Re\left\{ 1 + \frac{zs'(z)}{s(z)} \right\} > 0.
$$

An application of Lemma 1 [which is valid even though $N'(0)/D'(0) \neq 1$] yields

$$
Re\left\{ \frac{N(z)}{D(z)} \right\} > 0, \quad \text{or} \quad Re\left\{ \frac{zs'(z)}{s(z)} \right\} > 0, \quad \text{for} \quad z \in \Delta;
$$

and this together with

$$
\int_0^{2\pi} Re\left\{ \frac{re^{i\theta}s'(re^{i\theta})}{s(re^{i\theta})} \right\} \, d\theta = 2\pi \left\{ \frac{zs'(z)}{s(z)} \right\}_{z=0} = 4\pi, \quad 0 < r < 1,
$$

which follows from the mean-value theorem for harmonic functions, shows that $\sigma$ is 2-valent and starlike [1, p. 212]. One can show, furthermore, that $\sigma$ is convex [1] throughout the unit disk.
3. Theorems and proofs.

Proof of Theorem 1. For the function $S$, defined in Theorem 1, we have

$$\frac{zS'(z)}{S(z)} = \frac{zs(z) - \int_0^t s(t) \, dt}{\int_0^t s(t) \, dt} = \frac{z\sigma'(z) - \sigma(z)}{\sigma(z)}$$

and then differentiation of the numerator and the denominator of the last expression gives

$$\frac{[z\sigma'(z) - \sigma(z)]'}{\sigma'(z)} = \frac{z\sigma''(z)}{\sigma'(z)} = \frac{z\sigma'(z)}{\sigma(z)}.$$

An application of Lemma 1 and Lemma 2 completes the proof.

As an immediate consequence of Theorem 1 and the fact that

$$(3.1) \quad f \in \mathcal{K} \quad \text{if and only if} \quad zf' \in \mathcal{S}^*,$$

we have the following corollary.

Corollary 1.1. If $s \in \mathcal{S}^*$, then $f \in \mathcal{K}$ defines a member of $\mathcal{K}$.

Theorem 2. If $c \in \mathcal{K}$ and $C(z) = (2/z) \int_0^t c(t) \, dt$, then $C \in \mathcal{K}$.

Proof. Let $s(z) = zc'(z)$; then [by (3.1)] $s \in \mathcal{S}^*$. Let $S$ be defined as in Theorem 1, then

$$S(z) = \frac{2}{z} \int_0^t tc'(t) \, dt = \frac{2}{z} \left[ zc(z) - \int_0^t c(t) \, dt \right] = zC'(z).$$

Since $S$ is starlike, it follows from (3.1) that $C$ is convex. Theorem 2 can be proved directly from Lemma 1 by the method of Theorem 1.

Corollary 2.1. If $c \in \mathcal{K}$, $C(z) = (2/z) \int_0^t c(t) \, dt$ and $h(z) = 2c(z) - C(z)$, then $h \in \mathcal{S}^*$.

$f$ is in $\mathcal{C}$ if, and only if, there exists a function $g$ such that $g(z) = es(z)$, $s \in \mathcal{S}^*$, $|e| = 1$, and

$$\text{Re} \left\{ \frac{zf'(z)}{g(z)} \right\} > 0, \quad \text{for } z \text{ in } \Delta.$$
In this case, we say $f$ is close-to-convex with respect to the (starlike) function $g$.

**Theorem 3.** If $f$ is close-to-convex with respect to $g$,

$$ F(z) = \frac{2}{z} \int_0^z f(t) \, dt \quad \text{and} \quad G(z) = \frac{2}{z} \int_0^z g(t) \, dt, $$

then $F$ is close-to-convex with respect to $G$.

The proof is similar to that of Theorem 1.

In [3], J. Krzyż showed that the radius of close-to-convexity of every function in $S$ is greater than or equal to $r_0$, $0.8 < r_0 < 0.81$.

**Theorem 4.** If $f \in S$ and $F(z) = (2/z) \int_0^z f(t) \, dt$, then $F$ is schlicht (and close-to-convex) for $|z| < r$, $r \geq r_0$.

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**References**


University of Delaware