A MOMENT PROBLEM IN $L_1$ APPROXIMATION

CHARLES R. HOBBY$^1$ AND JOHN R. RICE$^2$

1. Introduction. The purpose of this paper is to show the existence of a solution to a certain moment problem which arises in the study of approximation in the $L_1$ norm

$$
\|f\| = \int_0^1 |f| \, d\mu.
$$

Let $n$ be a fixed, but arbitrary, integer and consider the sign function $s(A, x) = s(A)$ defined on $[0, 1]$ by

$$
s(A, x) = \begin{cases} 
+1 & x \in (a_i, a_{i+1}) \quad i \text{ even}, \\
0 & x = a_i, \\
-1 & x \in (a_i, a_{i+1}) \quad i \text{ odd}
\end{cases}
$$

where $A$ stands for the vector $(a_1, a_2, \ldots, a_n)$ and the convention is made that $a_i \leq a_{i+1}$, $a_0 = 0$, $a_{n+1} = 1$. Thus $s(A)$ is simply a step function taking on the values $\pm 1$ with at most $n$ sign changes. Let $\mu$ be a finite, nonatomic measure on $[0, 1]$ such that every $s(A)$ is measurable and let \{\phi_i\}_{i=1,2,\cdots,n}$ be $n$ functions integrable on $[0, 1]$. We may now state the

**Moment Problem.** Determine $A^*$ so that

$$
(MP) \quad \int \phi_is(A^*)d\mu = 0, \quad i = 1, 2, \cdots, n.
$$

In order to discuss $L_1$ approximation let $\mathcal{L}$ be a linear subspace of the space of integrable functions and suppose $\mathcal{L}$ is spanned by \{\phi_i\}. Set

$$
L(\alpha) = \sum_{i=1}^n \alpha_i \phi_i
$$

and let $f$ be an arbitrary integrable function not in $\mathcal{L}$. $L(\alpha^*)$ is said to be a best approximation to $f$ if

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$^2$ This work was done while this author was at the General Motors Research Laboratories.
for all $\alpha$.

Suppose that one knows a sign function $s(A)$ for which (MP) holds. One can then interpolate $f$ at $n$ canonical points $\{a_i\}$ and, in most but not all cases, obtain a best approximation. This procedure is elaborated upon in [9], [10], [12]. The main result of this paper guarantees the existence of a set of canonical points for any choice of $\{\phi_i(x)\}$. It also has application for nonlinear $L_1$ approximation [10].

The first connection in which the moment problem arises is as follows: Set

$$Z(\alpha) = \{x \mid f(x) = L(\alpha, x)\}.$$  

We have the following

**Characterization Theorem.** A necessary and sufficient condition for $L(\alpha^*)$ to be a best approximation to $f$ is that

$$\int L(\alpha) \text{ sgn } [f - L(\alpha^*)]d\mu \leq \int_{Z(\alpha^*)} |L(\alpha)| d\mu$$

holds for all $\alpha$.

It is difficult to determine the first correct statement and proof of this result, but it is given in full generality in [3]; for ordinary Lebesgue measure it follows from Theorem 1 of [13]. The hypothesis that $\mu$ is nonatomic and $s(A)$ is measurable is not required in this theorem.

There is an interesting special case of this theorem which has sometimes [1], [11] been mistaken for the characterization theorem itself. That is the following

**Corollary.** If the measure of $Z(\alpha^*)$ is zero, then a necessary and sufficient condition for $L(\alpha^*)$ to be a best approximation to $f$ is that

$$\int L(\alpha) \text{ sgn } [f - L(\alpha^*)]d\mu = 0$$

for all $\alpha$.

A secondary connection in which the moment problem arises is with the study of the **Haar Property**. It has recently been shown [7], [8] that no finite dimensional subspace has the Haar Property. Essential to both proofs (which are essentially the same) of this is the establishment of the solution of a certain moment problem by the use of a
II. The theorem and proof. The proof that the moment problem possesses a solution is outlined as follows. One considers the functions

\[ \Phi_i(A) = \int \phi_i s(A) d\mu \]

for \( i = 1, 2, \ldots, n \). The functions (4) are continuous functions of \( A \) since \( \mu \) is a finite nonatomic measure. One shows that the domain of definition of each \( \Phi_i \) may be identified with the \( n \)-ball

\[ B^n = \left\{ A \Big| \sum_{i=1}^{n} a_i^2 \leq 1 \right\} \]

One considers the transformation \( M \) of \( B^n \) into the \( n-1 \) sphere

\[ S^{n-1} = \left\{ A \Big| \sum_{i=1}^{n} a_i^2 = 1 \right\} \]

defined by

\[ M: A \to \frac{\left( \Phi_1(A), \Phi_2(A), \ldots, \Phi_n(A) \right)}{\sqrt{\sum [\Phi_i(A)]^2}} \]

This transformation is a continuous mapping unless all of the \( \Phi_i(A) \) are simultaneously zero for some \( A \), i.e., unless the moment problem has a solution. One may show that if \( A_1 \) and \( A_2 \) are pairs of antipodal points on \( B^n \) then

\[ \Phi_i(A_1) = - \Phi_i(A_2) \]

Thus \( M \) carries pairs of antipodal points of \( B^n \) into pairs of antipodal points of \( S^{n-1} \). It is known that such a transformation cannot be continuous.

The remainder of this section contains a detailed exposition of this proof.

**Lemma 1.** The function \( \Phi_i(A) \) is a continuous function of \( A \).

**Proof.** This follows from Holder's inequality and some well-known facts from measure theory [2]. In particular if \( \mu \) is nonatomic then the measure of a finite point set is zero.
The functions $\Phi_i(A)$ are defined in (4) on the $n$-simplex

$$K_n = \{ A \mid 0 \leq a_1 \leq \cdots \leq a_n \leq 1 \}.$$ 

A specific continuous identification is now constructed which identifies $K_n$ with $B^n$ such that the $\Phi_i$ are well defined on $B^n$ and, the essential point, in such a way that if $A_1$ and $A_2$ are pairs of antipodal points of $B^n$ then (6) holds.

Thus we inductively construct a mapping $\psi_n$ for each $n \geq 2$ which has the properties

(a) $\psi_n(0, x_2, \cdots, x_n) = -\psi_n(x_2, \cdots, x_n, 1)$ and each of these points is on the boundary of $B^n$.

(b) $\Phi_i(\psi_n(A))$ is well defined. That is to say if

$$\psi_n(A) = \psi_n(B) \quad \text{then} \quad \Phi_i(A) = \Phi_i(B).$$

It is easily verified from the specific form (4) of $\Phi_i(A)$ that property (b) follows from the following property

(b') if $\psi_n(x_1, \cdots, x_n) = \psi_n(y_1, \cdots, y_n)$ then for some $k$, $k$ of the $x_i$ are equal to $k$ of the $y_i$ and the remaining $x_i$ are equal to each other and the remaining $y_i$ are equal to each other.

We now show $\psi_2$ explicitly. Set $\psi_2(0, a) = (x, y)$ where $y = 2(a - \frac{1}{2})$ and $x = -\sqrt{1 - y^2}$. Set $\psi_2(a, 1) = -\psi_2(0, a), \psi_2(\frac{1}{2}, \frac{1}{2}) = (0, 0)$. For the remaining points in $K_2$, for $0 \leq t \leq 1$ set

$$\psi_2(t, \frac{1}{2}, \frac{1}{2}) = (1 - t)(0, a),$$

$$\psi_2[(\frac{1}{2}, \frac{1}{2}) + (1 - t)(a, 1)] = (1 - t)\psi_2(a, 1).$$

We note that every point in $K_2$ is of the form $t(\frac{1}{2}, \frac{1}{2}) + (1 - t)(0, a)$ or $t(\frac{1}{2}, \frac{1}{2}) + (1 - t)(a, 1)$ for $0 \leq t \leq 1$. Further $\psi_2(x_1, x_2) = \psi_2(y_1, y_2)$ if and only if $x_1 = x_2 = \frac{1}{2} - t$, $y_1 = y_2 = \frac{1}{2} + t$. Thus $\psi_2$ is a mapping of $K_2$ onto $B^2$ with properties (a) and (b').

We define the continuous mapping $\psi_{n+1}$ from $\psi_n$ as follows. Here $|\psi_n(x_1, \cdots, x_n)|$ denotes the usual Euclidean norm of $\psi_n(x_1, \cdots, x_n)$.

$$\psi_{n+1}(0, x_1, \cdots, x_n)$$

$$= (- \frac{1}{1 - |\psi_n(x_1, \cdots, x_n)|})^{1/2}, - \psi_n(x_1, \cdots, x_n),$$

$$\psi_{n+1}(x_1, \cdots, x_n, 1) = - \psi_{n+1}(0, x_1, \cdots, x_n),$$

$$\psi_{n+1}(\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}) = (0, 0, \cdots, 0),$$

$$\psi_{n+1}(t, \frac{1}{2}, \cdots, \frac{1}{2}) + (1 - t)(0, x_1, \cdots, x_n)$$

$$= (1 - t)\psi_{n+1}(0, x_1, \cdots, x_n), \quad 0 \leq t \leq 1,$$

$$\psi_{n+1}(t, \frac{1}{2}, \cdots, \frac{1}{2}) + (1 - t)(x_1, \cdots, x_n, 1)$$

$$= (1 - t)\psi_{n+1}(x_1, \cdots, x_n, 1), \quad 0 \leq t \leq 1.$$
Since \( \psi_n \) maps \( K_n \) onto \( B^n \) it follows from (7) and (8) that \( \psi_{n+1} \) is at least onto the boundary of \( B^{n+1} \) and hence it follows from (9), (10) and (11) that \( \psi_{n+1} \) maps all of \( K_{n+1} \) onto \( B^{n+1} \).

It is clear that \( \psi_{n+1} \) is continuous except possibly at the interface between the regions mapped by (10) and (11). Points in that interface are of the form \((a, x_1, \cdots, x_{n-1}, b)\) where \(a+b=1\). The mapping is well defined for these points if it is well defined for the points of the interface which map into the boundary of \( B^{n+1} \). These points are of the form \((0, x_1, x_2, \cdots, x_{n-1}, 1)\) and one may verify that (7) and (8) both map this point into the same point of \( B^{n+1} \).

That \( \psi_{n+1} \) has property (a) follows directly from (7) and (8).

To establish (b') suppose that \( \psi_{n+1}(x_1, \cdots, x_{n+1}) = \psi_{n+1}(y_1, \cdots, y_{n+1}) = B \). Suppose, for concreteness, that the first coordinate of \( B \) is negative. Then \( \psi_{n+1} \) is defined by (10). Thus

\[
B = (1 - t)\psi_{n+1}(0, u_1, \cdots, u_n) = (1 - s)\psi_{n+1}(0, v_1, \cdots, v_n)
\]

where

\[
(x_1, \cdots, x_{n+1}) = t(\frac{1}{2}, \cdots, \frac{1}{2}) + (1 - t)(0, u_1, \cdots, u_n),
\]

\[
(y_1, \cdots, y_{n+1}) = s(\frac{1}{2}, \cdots, \frac{1}{2}) + (1 - s)(0, v_1, \cdots, v_n).
\]

Since \(|\psi_{n+1}(0, u_1, \cdots, u_n)| = |\psi_{n+1}(0, v_1, \cdots, v_n)| = 1\) it follows that \(s = t\) and \(\psi_n(u_1, \cdots, u_n) = \psi_n(v_1, \cdots, v_n)\). By property (b') for \(\psi_n\) we have that \(k\) of the \(u_i\) equal \(k\) of the \(v_i\), while the remaining \(u_i\) are equal and the remaining \(v_i\) are equal. Since \(s = t\) it follows from (12) that \(\psi_{n+1} \) has property (b'). A similar argument applies if the first coordinate of \(B\) is positive.

The following lemma is well-known [4].

**Lemma 3.** There exists no continuous mapping of \( B^n \) into \( S^{n-1} \) such that pairs of antipodal points are mapped into pairs of antipodal points.

The main theorem may now be established.

**Theorem.** The moment problem (MP) has a solution for any set \( \{\phi_i\} \) of \( n \) integrable functions.

**Proof.** It has been established that the domain \( K_n, n \geq 2 \) of definition of \( \Phi_i(A) \) may be mapped onto \( B^n \) with properties (a) and (b). In particular, a pair \( A_1, A_2 \) of antipodal points satisfies (6). The mapping (5) is thus well defined and continuous unless

\[
\Phi_i(A) = 0, \quad i = 1, 2, \cdots, n.
\]

Since \( M \) takes \( B^n \) into \( S^{n-1} \) with pairs of antipodal points going into
pairs of antipodal points, it follows from Lemma 3 that \( M \) is not continuous and hence (13) is satisfied for some \( A \). The proof is trivial if \( n = 1 \). This concludes the proof.

It is of some interest to note [5] that if the \( \phi_i \) are continuous and form a Tchebycheff set, then \( s(A) \) is uniquely determined and must have exactly \( n \) sign changes.

References

10. ———, *On the computation of \( L_1 \) approximations by exponentials, rationals and other functions*, Math. Comp. 18 (1964), 390–396.

University of Washington and Purdue University