Proof. Let $X_1$ be the set of all $x$ in $X$ for which $|x| = 1$. Since each $G_j \subseteq G = \text{gp } X$, we conclude by Lemma 8 that each $G_j \subseteq \text{gp } X_1$, and hence that $G = \text{gp } X_1$. It follows from the irreducibility of $X$ that $X = X_1$.

This completes the proof of Grushko's Theorem.

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GENERALIZED FUNCTIONS OF SYMMETRIC MATRICES

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1. Introduction. In an abstract published in 1961 [4] we announced the following result:

Let $A$ be an $n$-square positive semi-definite matrix and assume that $A \geq S$ where $S$ is doubly stochastic. Then

\begin{equation}
\text{per}(A) \geq n! / n^n.
\end{equation}

The notation $A \geq S$ means $a_{ij} \geq s_{ij}$, $i, j = 1, \ldots, n$. A doubly stochastic (d.s.) matrix has non-negative entries and every row and column sum is 1. The permanent, per $(A)$, is the function defined by

\begin{equation}
\text{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{\sigma(i)}
\end{equation}

where the summation extends over the whole symmetric group of degree $n$, $S_n$.

In 1962 [3] we also proved that:

If $S$ is an $n$-square positive semi-definite symmetric matrix which is doubly stochastic in the extended sense then

\begin{equation}
\text{per}(S) \geq n! / n^n.
\end{equation}

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A matrix is d.s. in the extended sense if every row and column sum is 1; however, the elements need not all be non-negative. The inequality (1.3) constitutes what is currently known about a conjecture of van der Waerden that states:

$$\text{per } (S) \geq n!/n^n$$

for any $n$-square d.s. matrix $S$.

Observe that for the inequality (1.1) no assumption is made about $S$ being positive semi-definite, otherwise we obviously could get the result from (1.3). One might wonder that if a d.s. matrix $S$ exists for which $A \geq S$, then perhaps a positive semi-definite matrix $S_1$ exists which is d.s. in the extended sense and satisfies $A \geq S_1$. In other words one could hope to relax the d.s. condition in exchange for positive semi-definiteness. This is unfortunately not true. For, take

$$A = \begin{pmatrix}
\frac{1}{4} & 0 & \frac{3}{4} \\
0 & 1 & 0 \\
\frac{3}{4} & 0 & 3
\end{pmatrix}$$

which is obviously positive semi-definite. Clearly $A \geq S$ where

$$S = \begin{pmatrix}
\frac{1}{4} & 0 & \frac{3}{4} \\
0 & 1 & 0 \\
\frac{3}{4} & 0 & \frac{1}{4}
\end{pmatrix}$$

$S$ is not positive semi-definite. Thus we can try for an $S_1$ such that $A \geq S_1$, $S_1$ is d.s. in the extended sense, and $S_1$ is positive semi-definite. Set

$$S_1 = \begin{pmatrix}
a & c & 1 - (a + c) \\
c & b & 1 - (b + c) \\
1 - (a + c) & 1 - (b + c) & a + b + 2c - 1
\end{pmatrix}.$$

If $S_1$ is to be positive semi-definite then

$$0 \leq a(b + b + 2c - 1) - (1 - a - c)^2,$$

which simplifies to

(1.4) \hspace{1cm} (b + 1)a \geq (c - 1)^2.

If $A \geq S_1$ then $ab + a \leq \frac{3}{2}$ and $c \leq 0$, which are incompatible with (1.4).

Since the appearance of the original abstract we have done a substantial amount of work on generalized matrix functions as originally
defined by I. Schur \[7\]. Thus let $G$ be a subgroup of $S_n$ and let $\chi$ be a character (of arbitrary degree) of $G$. Following Schur we define the generalized matrix function $d_\chi$ by

\[
d_\chi(A) = \sum_{\sigma \in G} \chi(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}
\]

for any $n$-square matrix $A$. Clearly if $G = S_n$ and $\chi$ is the character of $G$ identically 1 then $d_\chi(A) = \text{per}(A)$. Schur \[7, \text{Theorem 1}\] has shown that $d_\chi(A) > 0$ for positive definite hermitian $A$. It follows that $d_\chi(A) \geq 0$ for positive semi-definite $A$. The purpose of this paper is to prove the following:

**Theorem.** Let $A$ be an $n$-square positive semi-definite real symmetric matrix and assume that $A \preceq S$ where $S$ is doubly stochastic. Then

\[
d_\chi(A) \geq \frac{m(\chi)}{n^n}
\]

where

\[
m(\chi) = \sum_{\sigma \in G} \chi(\sigma).
\]

For $G = S_n$ and $\chi \equiv 1$, $m = m(\chi) = n!$, and (1.6) specializes to (1.1). It is always true of course that $m(\chi)$ is either 0 or a positive integral multiple of the order of $G$. More precisely let $\chi = \chi_1 + \ldots + \chi_k$ be a representation of the character $\chi$ as a sum of irreducible characters. Then $m(\chi)/g$ is the number of $\chi_i$ in this representation which equal the trivial character 1. Here $g$ is the order of $G$. Thus for nontrivial irreducible $\chi$ our theorem reduces to Schur’s result.

2. **Preliminary results.** The proof of the inequality (1.6) depends on three theorems of interest in themselves.

**Theorem 1.** If $V$ and $W$ are arbitrary $n$-square complex matrices then

\[
|d_\chi(VW)|^2 \leq d_\chi(VV^*)d_\chi(W^*W).
\]

This result has appeared as a research announcement \[2\].

**Theorem 2.** If $A$ is a symmetric $n$-square matrix and $A \succeq 0$ (i.e., $a_{ij} \geq 0$, $i, j = 1, \ldots, n$) then there exists a unique diagonal matrix $D = \text{diag}(d_1, \ldots, d_n)$, $d_i > 0$, $i = 1, \ldots, n$, such that $DAD$ is d.s.

The authors knew this theorem at the time of the announcement \[4\] and shortly thereafter a constructive proof was found by Maxfield and...
Minc [6]. Independently Sinkhorn [8], [9] proved a closely related result.

**Theorem 3.** Let $D$ be the diagonal matrix described in Theorem 2. Moreover, assume that $A \geq S$ where $S$ is d.s. Then

$$\prod_{i=1}^{n} d_i \leq 1.$$  

**Proof.** For any $n$-square matrix $X$ let $r_i(X)$ denote the $i$th row sum of $X$. Now $A \geq S$ implies that $DAD \geq DSD$ and since $DAD$ is d.s. we have

$$1 = r_i(DAD) \geq r_i(DSD) = d_i \sum_{j=1}^{n} s_{ij}d_j.$$  

Thus

$$\prod_{i=1}^{n} d_i \prod_{i=1}^{n} \sum_{j=1}^{n} s_{ij}d_j.$$  

Let

$$g(X) = \prod_{i=1}^{n} \left( \sum_{j=1}^{n} x_{ij}d_j \right)^{1/n}$$

where $X$ is an arbitrary d.s. matrix. It is a well-known result of Birkhoff [1] that the totality $\Omega_\pi$ of $n$-square d.s. matrices is a convex polyhedron with the permutation matrices as vertices. If $X, Y \in \Omega_\pi$, $0 \leq \theta \leq 1$, then the Hölder inequality implies that

$$g(\theta X + (1 - \theta) Y) = \prod_{i=1}^{n} \left( \sum_{j=1}^{n} (\theta x_{ij} + (1 - \theta) y_{ij})d_j \right)^{1/n}$$

$$= \prod_{i=1}^{n} \left( \theta \sum_{j=1}^{n} x_{ij}d_j + (1 - \theta) \sum_{j=1}^{n} y_{ij}d_j \right)^{1/n}$$

$$\geq \theta g(X) + (1 - \theta) g(Y).$$

Hence $g$ is concave on $\Omega_\pi$ and assumes its minimum on a permutation matrix. The value of $g$ on any permutation matrix is just $(\prod_{i=1}^{n} d_i)^{1/n}$. From (2.3) we can conclude that

$$1 \geq \prod_{i=1}^{n} (g(X))^n \geq \prod_{i=1}^{n} d_i^2,$$

and (2.2) follows.
3. **Proof of the Theorem.** In [5] it was proved that if \( R \) is any d.s. positive semi-definite matrix then the positive semi-definite determination of the square root of \( R \), \( R^{1/2} \), is d.s. in the extended sense.

In Theorem 1 let \( V = R^{1/2} \) and \( W = J_n \), the matrix with every entry \( 1/n \). Then \( R^{1/2}J_n = J_n \), \( J_n^2 = J_n \), and thus

\[
\left| d_x(J_n) \right|^2 \leq d_x(R)d_x(J_n).
\]

It is known from Schur’s theorems on the \( d_x \) function that \( d_x(R) \geq 0 \). \((R \text{ is positive semi-definite.})\) Hence whether \( m(x) \) is 0 or not we have

\[
(3.1) \quad d_x(R) \geq d_x(J_n) = m(x)/n^n.
\]

We can assume by continuity that \( A > 0 \) in proving (1.6). By Theorem 2 choose a diagonal matrix \( D = \text{diag} (d_1, \cdots, d_n) \), \( d_i > 0 \), \( i = 1, \cdots, n \) for which \( DAD = R \) is d.s. Then

\[
(3.2) \quad d_x(DAD) = d_x(R) \geq m(x)/n^n.
\]

But by Theorem 3

\[
(3.3) \quad d_x(DAD) = \prod_{i=1}^{n} d_i^2 d_x(A) \leq d_x(A).
\]

The inequalities (3.2) and (3.3) complete the proof.

**References**


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