AN EXACT SEQUENCE IN GALOIS COHOMOLOGY

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Let $A$ be an integrally closed noetherian domain with the quotient
field $F$. The group of divisors of $A$ is the free abelian group generated
by nonzero minimal prime ideals of $A$ and is denoted by $D(A)$. This
is canonically isomorphic to the group gotten from the set of all re-
flexive $A$-ideals (including fractional ideals) under the rule $a \cdot b
= (a \cdot b)^\star$ where $\epsilon^\star = \{a \in F \mid a \subset A\} = \text{Hom}_A(\epsilon, A)$. The divisor class
group of $A$ denoted by $C(A)$ is the factor group of $D(A)$ by the prin-
cipal divisors, i.e. it is defined by the exact sequence

$$0 \to U(F)/U(A) \to D(A) \to C(A) \to 0$$

where $D(a) = \sum_p \nu_p(a) \mathfrak{p}$ with $(pA_p)^{\nu_p(a)} = aA_p$. We observe that $A$ is
a unique factorization domain if and only if $C(A) = 0$, i.e. if and only
if $U(F)/U(A) \to D(A)$ is an isomorphism.

Now let $S \supset R$ be an integral extension of an integrally closed
noetherian domain, whose quotient field $L \supset K$ is a separable exten-
sion of finite degree. Then we obtain the canonical map $i: D(R) \to D(S)$
given by $\sum_p \nu_p \to \sum_p \nu_p (\sum_{p|\mathfrak{p}} e(\mathfrak{p}) \mathfrak{p})$ where $e(\mathfrak{p})$ is the ramification
index of $\mathfrak{p}$ in $S \supset R$, i.e. $\mathfrak{p}S = (\mathfrak{p}S')^e(\mathfrak{p})$. Since the map $i$ sends the
principal divisors to principal divisors, it induces the map $i: C(R)
\to C(S)$. We denote the kernel of $i$ by $C(S/R)$. Thus $C(S/R)$ is the sub-
group of $C(R)$ consisting of those divisor classes which become prin-
cipal under the extension $S \supset R$. Now let the quotient field extension
$L \supset K$ be Galois with the Galois group $G$. As customary we denote
$H^n(G, U(L)), H^n(G, U(S))$ by $H^n(L/K), H^n(S/R)$ respectively. The
main purpose of this short note is to prove:

**Theorem.** Let $S \supset R$ be an integral extension of an integrally closed
noetherian domain whose quotient field extension $L \supset K$ is Galois with
the Galois group $G$. Then we have the exact sequence

$$0 \to C(S/R) \to H^1(S/R) \to D(S)/iD(R) \to C(S)/iC(R) \to$$

$$\to H^2(S/R) \to \bigcap H^2(S/R_\mathfrak{p}) \to H^1(G, C(S)) \to H^3(S/R) \to$$

Received by the editors May 14, 1964.

1 This work was partially supported by NSF GP-218.

2 For any commutative ring $A$, we denote by $U(A)$ the group of invertible ele-
ments in $A$.  

837
where \( \cap_p H^2(S_p/R_p) = \cap_p \text{Im}(H^3(S_p/R_p) \to H^2(L/K)) \), \( p \) running through all nonzero minimal primes of \( R \).

**Remark.** A somewhat similar exact sequence related to the Brauer groups in the case when \( S \supset R \) is unramified was obtained in [2], [3].

**Proof.** Firstly we observe that \( H^1(G, D(S)) = 0 \). Indeed, if we fix, for each nonzero minimal prime \( p \) in \( R \), a nonzero minimal prime \( \mathfrak{p} \) in \( S \) lying above \( p \), and if we denote by \( G_{\mathfrak{p}} \) the decomposition subgroup of \( \mathfrak{p} \) over \( p \), then \( D(S) \cong \sum_p Z[G_{\mathfrak{p}}] \otimes_{Z[G]} Z \) as \( G \)-modules, where \( p \) runs through all nonzero minimal primes of \( R \). Consequently \( H^*(G, D(S)) = \sum_p H^*(G_{\mathfrak{p}}, Z) \) and in particular we have \( H^1(G, D(S)) = 0 \). Now for each minimal prime \( p \) in \( R \), \( S_p \) is a unique factorization domain and hence \( 0 \to U(S_p) \to U(L) \to D(S_p) \to 0 \) is an exact sequence of \( G \)-modules. Therefore \( 0 \to H^2(S_p/R_p) \to H^2(L/K) \to H^2(G, D(S_p)) \) is exact and hence we obtain the exact sequence

\[
0 \to \bigcap_p H^2(S_p/R_p) \to H^2(L/K) \to H^2(G, D(S)).
\]

The exact sequence of \( G \)-modules \( 0 \to U(L) \to U(S) \to D(S) \to C(S) \to 0 \) together with \( H^1(G, D(S)) = 0 \) gives us the exact sequences

\[
(2) \quad 0 \to (U(L)/U(S))^a \to D(S)^a \to C(S)^a \to H^1(G, U(L)/U(S)) \to 0,
\]

\[
(3) \quad 0 \to H^1(G, C(S)) \to H^1(G, U(L)/U(S)) \to H^2(G, D(S)).
\]

In turn the exact commutative diagram

\[
\begin{array}{ccc}
0 & \to & U(K)/U(R) \\
\downarrow & & \downarrow \\
0 & \to & (U(L)/U(S))^a \\
\end{array}
\]

yields the exact sequence

\[
0 \to C(S/R) \to (U(L)/U(S))^a/(U(K)/U(R)) \to D(S)^a/iD(R) \to C(S)^a/iC(R) \to H^1(G, U(L)/U(S)) \to 0.
\]

On the other hand, the exact sequence \( 0 \to U(S) \to U(L) \to U(L)/U(S) \to 0 \) together with Hilbert's Theorem 90 gives us the exact sequences

\[
(5) \quad 0 \to U(R) \to U(K) \to (U(L)/D(S))^a \to H^1(S/R) \to 0,
\]

\[
(6) \quad 0 \to H^1(G, U(L)/U(S)) \to H^2(S/R) \to H^2(L/K) \to \cdots.
\]

Now the exact commutative diagram
0 \rightarrow H^1(G, U(L)/U(S)) \rightarrow \\
H^2(S/R) \rightarrow H^2(L/K) \rightarrow H^2(G, U(L)/U(S)) \rightarrow H^2(S/R) \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
0 \rightarrow H^2(G, D(S)) \rightarrow H^2(G, D(S)) \rightarrow 0

Together with (1) and (3) yields the exact sequence

0 \rightarrow H^1(G, U(L)/U(S)) \rightarrow H^2(S/R) \rightarrow \bigcap_b H^2(S_b/R_b) \\
(7) \rightarrow H^1(G, C(S)) \rightarrow H^2(S/R).

On the other hand, (4) and (5) gives us

0 \rightarrow C(S/R) \rightarrow H^1(S/R) \rightarrow D(S)^a/iD(R) \rightarrow C(S)^a/iC(R) \\
(8) \rightarrow H^1(G, U(L)/U(S)) \rightarrow 0.

Connecting (7) and (8) we obtain the desired exact sequence.

When \( S \supset R \) is unramified, we can relate the 2-dimensional cohomology group with the Brauer group. We denote by \( B(S/R) \) the kernel of the canonical map \( B(R) \rightarrow B(S) \), where \( B( ) \) denotes the Brauer group. Then \( H^2(S/R) = B(S/R) \) if \( S \supset R \) is unramified and \( R \) is a local domain [1] and thus we obtain:

**Corollary 1.** Let \( S \supset R \) be unramified. Then we have the exact sequence

\[
0 \rightarrow H^1(S/R) \rightarrow C(R) \rightarrow C(S)^a \rightarrow H^2(S/R) \rightarrow \bigcap_b B(S_b/R_b) \\
\rightarrow H^1(G, C(S)) \rightarrow H^2(S/R).
\]

**Proof.** If \( S \supset R \) is unramified, then \( D(S)^a = iD(R) \) and \( \bigcap_b H^2(S_b/R_b) = \bigcap_b B(S_b/R_b) \).

If we further assume \( R \) to be regular, our exact sequence coincides with the exact sequence in [2], [3].

**Corollary 2.** Let \( S \supset R \) be unramified. If \( R \) is regular, we have the exact sequence

\[
0 \rightarrow H^1(S/R) \rightarrow C(R) \rightarrow C(S)^a \rightarrow H^2(S/R) \rightarrow B(S/R) \\
\rightarrow H^1(G, C(S)) \rightarrow H^3(S/R).
\]

**Proof.** We must show that \( \bigcap_b B(S_b/R_b) = B(S/R) \). Since one side inclusion is clear [1], it suffices to show that \( \bigcap_b B(S_b/R_b) \subset B(S/R) \), i.e. \( \text{Ker}(H^2(L/K) \rightarrow H^2(G, D(S)) \subset B(S/R) \). Now \( S \), being unramified over a regular domain \( R \), is also regular and hence is a local unique factorization domain. Consequently \( D(S) \) is nothing but the group of
invertible $S$-ideals. Let $\alpha \in \text{Ker}(H^3(L/K) \to H^3(G, D(S)))$, and let \( \{a_{s, r}\} \) be a 2-cocycle representing $\alpha$. This means that there exists a set \( \{A_s\} \) of invertible $S$-ideals indexed by $G$ such that $a_{s, r}A_s A_r^{-1} = S$, i.e. $a_{s, r}A_s A_r^* = A_{s, r}$. (We may assume that $A_1 = S$.) Now let $\Gamma$ be the central simple $K$-algebra associated with the 2-cocycle \( \{a_{s, r}\} \), i.e. $\Gamma = \sum_s L\mu_s$ (direct sum) with the multiplication rule: $(x_s u_s)(x_r u_r) = x_s x_r a_{s, r} u_{s, r}$. Then $\Lambda = \sum_s A_s \mu_s$ which is a subset of $\Gamma$ is stable under the multiplication since $A_s u_s A_r u_r = A_s A_r^* a_{s, r} u_{s, r} = A_{s, r} u_{s, r}$. Thus $\Lambda$ is an order over $R$ in $\Gamma$, and is projective as an $R$-module. Now consider the canonical map $S \otimes_R \Lambda \to \text{Hom}_S(\Lambda, \Lambda)$ given by $(s \otimes \lambda)(x) = sx\lambda$. Since $\alpha \in \text{Ker}(H^3(L/K) \to H^3(G, D(S)) = \bigcap_p H^3(S_p/R_p)$, it follows that $S_p \otimes \Lambda_p \to \text{Hom}_{S_p}(\Lambda_p, \Lambda_p)$ is an isomorphism for all non-zero minimal primes $p$ in $R$. Consequently the canonical map $S \otimes_R \Lambda \to \text{Hom}_S(\Lambda, \Lambda)$ is an isomorphism since both sides are $R$-projective modules of the same rank. Therefore $\Lambda$ is an $R$-separable order in $\Sigma$ with $S$ as a splitting ring, and this completes our proof.

Bibliography


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