A REMARK ON THE GENERAL SUMMABILITY THEOREM

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1. Introduction. The following theorem is known as the general summability theorem in the theory of Fourier integrals (see Titchmarsh [4, p. 28]).

**Theorem A.** Suppose that a function $K(x, y, \delta)$ of $y$ belongs to $L^1(-\infty, \infty)$ and satisfies, for a fixed $x$,

\begin{align*}
(1.1) & \quad K(x, y, \delta) = O(1/\delta), \quad \text{for} \quad |y - x| \leq \delta, \\
(1.2) & \quad = O(\delta^\alpha/|x - y|^{1+\alpha}), \quad \text{for} \quad |x - y| > \delta,
\end{align*}

for some positive $\alpha$ and

\begin{align*}
(1.3) & \quad \lim_{\delta \to 0} \int_x^\infty K(x, y, \delta) \, dy = 1/2, \\
(1.4) & \quad \lim_{\delta \to 0} \int_{-\infty}^x K(x, y, \delta) \, dy = 1/2.
\end{align*}

Let $f(y)/(1 + |y|^{\alpha+1})$ belong to $L^1(-\infty, \infty)$. Then

\begin{align*}
(1.5) & \quad \lim_{\delta \to 0} \int_{-\infty}^\infty K(x, y, \delta)f(y) \, dy = (1/2)\{\phi(x) + \psi(x)\},
\end{align*}

wherever

\begin{align*}
(1.6) & \quad \int_0^h |f(x + t) - \phi(x)| \, dt = O(h) \\
(1.7) & \quad \int_0^h |f(x - t) - \psi(x)| \, dt = O(h)
\end{align*}

as $h \to 0 +$.

This implies Fejér's integral theorem and other kinds of ordinary summability theorems. However, Theorem A is not true if $\alpha=0$ and as a matter of fact, it does not imply the Fourier single integral theorem. In this paper we shall prove a theorem which corresponds to the

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case when \( \alpha = 0 \) and implies as a particular case the ordinary Fourier single integral theorem.

2. **Theorem and its proof.** We shall prove the following

**Theorem.** Suppose that for a fixed \( x \) the improper integral
\[
\int_{-\infty}^{\infty} K(x, y, \delta) \, dy
\]
exists. Let for fixed \( x \) and \( \eta > 0 \),
\[
(2.1) \quad K(x, y, \delta) = O(\left| x - y \right|^{-1}), \quad \text{for} \quad \left| x - y \right| > \eta > 0,
\]
where \( O \) is independent of \( \delta \) (but may depend on \( \eta \)).

Suppose further that
\[
(2.2) \quad \lim_{\delta \to 0} \int_{x}^{\infty} K(x, y, \delta) \, dy = p,
\]
\[
(2.3) \quad \lim_{\delta \to 0} \int_{-\infty}^{x} K(x, y, \delta) \, dy = 1 - p,
\]
where \( 0 < p < 1 \);

\[
(2.4) \quad \lim_{\delta \to 0} \int_{\beta}^{\infty} K(x, y, \delta) \, dy = 0, \quad \text{for every constant } \beta > x,
\]
\[
(2.5) \quad \lim_{\delta \to 0} \int_{-\infty}^{\alpha} K(x, y, \delta) \, dy = 0, \quad \text{for every constant } \alpha < x;
\]
\[
(2.6) \quad \lim_{\delta \to 0} \int_{\alpha}^{\beta} (y - x)K(x, y, \delta) \, dy = 0,
\]
for every pair \( \alpha, \beta \) of constants such that \( x < \alpha < \beta \) or \( \alpha < \beta < x \); and

\[
(2.7) \quad \int_{x}^{\beta} K(x, y, \delta) \, dy = O(1)
\]
for every constant \( \beta \), \( O \) being independent of \( \delta \).

Let \( f(y)/(1 + |y|) \in L^1(-\infty, \infty) \) and let \( f(y) \) be of bounded variation in a neighborhood of \( x \). Then we have

\[
(2.8) \quad \lim_{\delta \to 0} \int_{-\infty}^{\infty} f(y)K(x, y, \delta) \, dy = pf(x + 0) + (1 - p)f(x - 0).
\]

Note that the integral on the left-hand side is absolutely convergent, because using (2.1) we have

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* I missed the conditions (2.4) and (2.5) in the original form of the theorem. I am obliged to Professor S. Bochner for calling this point to my attention.
for a large $A$. The same thing is true for $\int_{-A} f$.

**Proof of the theorem.** It is sufficient to prove that

$$\lim_{t \to 0} \left\{ \int_{x}^{x+t} K(x, y, \delta) f(y) \, dy \right\} = 0,$$

together with a similar result in terms of $f_{+\infty}$ and $f_{-\infty}$.

We may suppose without loss of generality that $f(y)$ is nondecreasing in a right-hand neighborhood of $x$. For any given positive $\epsilon$ we choose $\eta$ such that

$$\left| f(y) - f(x + 0) \right| < \epsilon, \quad \text{for } x \leq y \leq x + \eta,$$

and we write the expression in (2.9) in the following way.

$$I = \int_{x}^{x+\eta} K(x, y, \delta) \left\{ f(y) - f(x + 0) \right\} \, dy$$
$$+ \int_{x+\eta}^{x+\eta+1} K(x, y, \delta) \, dy - f(x + 0) \int_{x+\eta}^{\infty} K(x, y, \delta) \, dy$$
$$= I_1 + I_2 + I_3,$$

say.

For every fixed $\eta > 0$, we have, because of (2.4),

$$\lim_{\delta \to 0} I_3 = 0.$$

The second mean value theorem shows that

$$\left| I_1 \right| \leq \epsilon \int_{x+\xi}^{x+\eta} K(x, y, \delta) \, dy$$

for some $0 < \xi < \eta$. Hence by (2.7), there exists a constant $C$ independent of $\delta$ such that

$$\left| I_1 \right| \leq C \epsilon.$$

In order to handle $I_2$, we split it into two parts

$$I_2 = \int_{x+\eta}^{x+\eta+1} K(x, y, \delta) f(y) \, dy + \int_{x+\eta}^{\infty} K(x, y, \delta) f(y) \, dy$$
$$= I_{21} + I_{22}, \quad \eta < A.$$
\[ |I_{22}| \leq \int_{x+A}^{\infty} \frac{|f(y)|}{y-x} (y-x) |K(x,y,\delta)| \, dy \leq C_1 \int_{x+A}^{\infty} \frac{|f(y)|}{y-x} \, dy, \]

where \( C_1 \) is a constant and the integral on the right-hand side may be made as small as desired by taking \( A \) large, since \( f(y)/(1+|y|) \in L^1(-\infty, \infty) \). Hence we may write

\[ (2.13) \quad |I_{22}| \leq C_2 \epsilon, \]

where \( C_2 \) is a constant.

Now we choose a step-function \( g(y) \) in \((x+\eta, x+A)\), with a finite number of jumps, in such a way that

\[ \int_{x+\eta}^{x+A} \left| \frac{f(y)}{y-x} - g(y) \right| \, dy < \epsilon \]

for fixed \( A \) and \( \eta \). We then write

\[ I_{21} = \int_{x+\eta}^{x+A} (y-x) K(x,y,\delta) \left( \frac{f(y)}{y-x} - g(y) \right) \, dy \]

\[ + \int_{x+\eta}^{x+A} (y-x) K(x,y,\delta) g(y) \, dy. \]

Then

\[ |I_{21}| \leq C \int_{x+\eta}^{x+A} \left| \frac{f(y)}{y-x} - g(y) \right| \, dy + \int_{x+\eta}^{x+A} g(y)(y-x) K(x,y,\delta) \, dy \]

so that

\[ (2.14) \quad |I_{21}| \leq C \epsilon + \int_{x+\eta}^{x+A} g(y)(y-x) K(x,y,\delta) \, dy. \]

We shall show that the last integral converges to zero as \( \delta \to 0 \). To do this it is sufficient to prove that

\[ (2.15) \quad \int_{x}^{\beta} (y-x) K(x,y,\delta) \, dy \to 0, \quad \text{for } x+y \leq \alpha < \beta \leq x + A \]

which is no more than the condition (2.6).

Combining (2.11), (2.12), (2.13), (2.14) and (2.15) we have

\[ \limsup_{\epsilon \to 0} |I| \leq (2C + C_2) \epsilon. \]

The same thing is true for (2.9) with \((-\infty, x)\) and \( f(x-0) \). This proves the theorem.
3. Remarks. If $K(x, y, \delta) = \lambda K(\lambda(y-x))$ with $\delta = 1/\lambda$ and $K(x)$ is an even function, then (2.8) becomes

\begin{equation}
\lim_{\lambda \to \infty} \int_{-\infty}^{\infty} (x + \frac{u}{\lambda}) K(u) \, du = \frac{1}{2} \{f(x + 0) + f(x - 0)\}.
\end{equation}

The theorem states that (3.1) is true if $f(u)/(1+|u|) \in L(\infty, \infty)$, $f(u)$ is a function of bounded variation in a neighborhood of $x$ and $K(u)$ satisfies the conditions that

\begin{align*}
(3.2) & \int_{-\infty}^{\infty} K(u) \, du = 1, \\
(3.3) & \lambda \int_{\beta}^{\infty} K(\lambda u) \, du \to 0 \quad \text{as} \quad \lambda \to \infty, \quad \text{for every} \quad \beta > 0, \\
(3.4) & \lim_{\lambda \to \infty} \lambda \int_{\beta}^{\alpha} u K(\lambda u) \, du = 0, \quad \text{if} \quad \beta > \alpha > 0 \quad \text{or} \quad \alpha < \beta < 0, \\
(3.5) & \int_{0}^{\beta} u K(u) \, du = O(1), \quad \text{for every} \quad \beta, \\
\text{and} & \\
(3.6) & u K(u) = O(1).
\end{align*}

Since the function $\sin u/(\pi u)$ satisfies above conditions, (3.1) gives us the Fourier single integral theorem. As to the above form (3.1) of summability theorem, one may refer to Bochner [1], Bochner-Chandrasekharan [2] and Bochner-Izumi [3]. It will be noted that the proof given in this paper is just a generalization of the proof of the Fourier single integral theorem. $I_1$ and $I_2$ have been handled in the usual manner. In proving that $I_2$ converges to zero with the Dirichlet integral, the Riemann-Lebesgue lemma is usually used, but the method employed here is known to be applicable in proving the Riemann-Lebesgue lemma as observed by Bochner and Chandrasekharan [1, p. 4].

REFERENCES


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