ON UNITARY EQUIVALENCE OF HILBERT-SCHMIDT OPERATORS

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1. **Introduction.** In the study of the unitary equivalence problem for (bounded) operators on Hilbert space, it is frequently useful to know conditions under which *-algebra isomorphisms between certain operator algebras are implemented by a unitary operator on the underlying Hilbert space.

In particular, one of the authors proved a theorem of this nature [**3**, Theorem 2] concerning von Neumann (v.N.) subalgebras of a finite v.N. algebra of type I, and then used this result as a central tool in setting forth a complete set of unitary invariants for the class of n-normal operators [**3**, Corollary 3.1].

On the other hand, the other author set forth a complete set of unitary invariants for Hilbert-Schmidt operators [**1**, Theorem 4], but in doing so used only operator theoretic techniques. Thus while [**1**, Theorem 4] is easily seen to be the natural analogue for Hilbert-Schmidt operators of [**3**, Corollary 3.1], there was no natural analogue for Hilbert-Schmidt operators of the result [**3**, Theorem 2].

It is the purpose of this note to prove the natural analogue of [**3**, Theorem 2] for *-subalgebras of the algebra of Hilbert-Schmidt operators, and then to deduce [**1**, Theorem 4] from this result anew. This provides a proof for [**1**, Theorem 4] that is algebraic in character and somewhat simpler than the original proof in [**1**]. It seems quite interesting to us that this close parallelism exists between the results of [**3**] and the corresponding results for the Hilbert-Schmidt class.

2. We begin with some preparatory definitions and notation. Let $\mathcal{H}$ be any separable Hilbert space. We follow Schatten [**5**] by adopting the notation (τc) and (σc) for the *-algebras of trace-class and Hilbert-Schmidt operators on $\mathcal{H}$, respectively. If (σc) is assigned the Schmidt norm $\| \ \|_\sigma$, then (σc) becomes a Banach *-algebra, and in fact, an H^*-algebra [**4**]. (It is the known structure theory for H^*-algebras that makes our program possible.) The trace of an operator $A \in (\tau c)$ is denoted by $t(A)$. The reader is reminded that $\|A\|_\sigma^2 = t(A^*A)$ and that an operator $A \in (\tau c)$ if and only if $A = BC$ where $B, C \in (\sigma c)$.

Let W denote the free multiplicative semi-group on the symbols x and y, and let W' denote the set obtained by removing from W

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the two words x and y of length one. Words in W are denoted by $w(x, y)$, and the set of all polynomials $p(x, y)$ which are (finite, complex) linear combinations of words $w(x, y) \in W$ is denoted by P. Because of the fact that an operator $A \in (\tau c)$ has the same trace as any operator of the form $A \oplus 0$ (and we wish to discuss unitary equivalence as opposed to isometric equivalence), the following definitions are pertinent. For any operator A, let $\mathfrak{M}$ be the largest subspace of $\mathfrak{H}$ which reduces A and which has the property that the restriction of A to $\mathfrak{M}$ is the zero operator. (It is easy to see that $\mathfrak{M}$ is the intersection of the null spaces of A and A^*.) Denote by $\delta(A)$ the dimension of $\mathfrak{M}$, where the dimension of the subspace $\{0\}$ is taken to be zero. Also, if $\mathfrak{A}$ is any $*$-subalgebra of (σc), denote by $\mathfrak{M}(\mathfrak{A})$ the subspace $\bigcap_{A \in \mathfrak{A}} \mathfrak{M}(A)$ of $\mathfrak{H}$, and write $\delta(\mathfrak{A}) = \dim \mathfrak{M}(\mathfrak{A})$.

We now quote [**1**, Theorem 4].

THEOREM A. *Let A be any operator in (σc). Then A is unitarily equivalent to an operator B acting on $\mathfrak{H}$ if and only if $B \in (\sigma c)$, $\delta(A) = \delta(B)$, and $t[w(A, A^*)] = t[w(B, B^*)]$ for each $w(x, y) \in W'$.*

Before stating the analogue of [**3**, Theorem 2] that we shall use to prove Theorem A anew, we obtain the following lemma.

LEMMA 2.1. *Let $\boldsymbol{A}$ be any $*$-subalgebra of (σc) which is closed in the Schmidt norm. Then every $A \in \boldsymbol{A} \cap (\tau c)$ can be written $A = BC$ where $B, C \in \boldsymbol{A}$.*

PROOF. $\boldsymbol{A}$ is a semi-simple H^*-algebra along with (σc), and we consider first the case in which $\boldsymbol{A}$ is simple. Then [**4**, Theorem 4.10.32] $\boldsymbol{A}$ contains a complete set of matrix units $\{E_{\lambda,\mu}\}_{\lambda,\mu \in \Lambda}$ and consists exactly of the operators of the form

$$\sum_{\lambda,\mu} a_{\lambda\mu} E_{\lambda\mu}$$

where

$$\sum_{\lambda,\mu} |a_{\lambda\mu}|^2 < \infty;$$

the foregoing operator sum converges in the Schmidt norm. The projections $E_{\lambda\lambda} \in \boldsymbol{A}$ must be n-dimensional ($n < \infty$), and thus $\boldsymbol{A}$ is spatially the direct sum of an n-fold copy of the Hilbert-Schmidt class on some Hilbert space and an appropriate sized zero operator. It is easily seen from this and known facts about Hilbert-Schmidt operators that if $A \in \boldsymbol{A} \cap (\tau c)$ and $A = UP$ is the polar decomposition of A, then $P^{1/2} \in \boldsymbol{A}$ and $UP^{1/2} \in \boldsymbol{A}$. This takes care of the case in which

$\boldsymbol{A}$ is simple. In the general case, according to the known structure theorem [**4**, Theorem 4.10.31] there is a family $\{\boldsymbol{A}_\gamma\}_{\gamma\in\Gamma}$ of simple, mutually annihilating H^*-subalgebras $\boldsymbol{A}_\gamma$ of $\boldsymbol{A}$ such that $\boldsymbol{A}$ consists exactly of all operators $B=\sum_\gamma\oplus B_\gamma$ where $B_\gamma\in\boldsymbol{A}_\gamma$ and $\|B\|_\sigma^2 = \sum_\gamma \|B_\gamma\|_\sigma^2<\infty$. If $A=\sum_\gamma\oplus A_\gamma\in\boldsymbol{A}\cap(\tau c)$, and $A=UP$ is the polar decomposition of A, it is clear that $P=\sum_\gamma\oplus P_\gamma$ where $P_\gamma=(A_\gamma^*A_\gamma)^{1/2}$. Since $P\in(\tau c)$ along with A, $t(P)=\sum_\gamma t(P_\gamma)<\infty$, and this implies that $t(P_\gamma)<\infty$ for each $\gamma\in\Gamma$. Thus each $A_\gamma\in(\tau c)\cap\boldsymbol{A}_\gamma$, and using the result already obtained for simple algebras, we can write each A_γ in polar form $A_\gamma=U_\gamma P_\gamma$ where $U_\gamma P_\gamma^{1/2}$, $P_\gamma^{1/2}\in\boldsymbol{A}_\gamma$. Setting $B=\sum_\gamma\oplus U_\gamma P_\gamma^{1/2}$ and $C=\sum_\gamma\oplus P_\gamma^{1/2}$, we obtain $\|B\|_\sigma^2=\sum_\gamma\|U_\gamma P_\gamma^{1/2}\|_\sigma^2 = \sum_\gamma t(P_\gamma^{1/2}U_\gamma^*U_\gamma P_\gamma^{1/2})=\sum_\gamma t(P_\gamma)=t(P)<\infty$; similarly $\|C\|_\sigma^2<\infty$. Thus, B, $C\in\boldsymbol{A}$ and $A=BC$.

The following theorem is the analogue for *-subalgebras of (σc) of [**3**, Theorem 2].

THEOREM B. *Suppose that $\boldsymbol{A}$ and $\boldsymbol{B}$ are *-subalgebras of (σc) which are closed in the Schmidt norm and which satisfy $\delta(\boldsymbol{A})=\delta(\boldsymbol{B})$. Suppose also that ψ is a *-algebra isomorphism from $\boldsymbol{A}$ onto $\boldsymbol{B}$. Then there is a unitary operator U on $\mathcal{H}$ satisfying $\psi(A)=UAU^*$ for each $A\in\boldsymbol{A}$ if and only if $t(A)=t(\psi(A))$ for each $A\in\boldsymbol{A}\cap(\tau c)$.*

REMARK. It follows from Lemma 2.1 that if $A\in\boldsymbol{A}\cap(\tau c)$ then $A=BC$ where B, $C\in\boldsymbol{A}$. Thus $\psi(A)=\psi(B)\psi(C)\in\boldsymbol{B}\cap(\tau c)$, so that $t(\psi(A))$ is meaningful.

PROOF OF THEOREM B. If ψ is implemented by a unitary operator, then the fact that $t(A)=t(\psi(A))$ for $A\in\boldsymbol{A}\cap(\tau c)$ is a consequence of the fact that $t(\cdot)$ is a unitary invariant. Thus we turn to the proof of the other half of the theorem. Consider first the case that $\boldsymbol{A}$ is a simple H^*-subalgebra of (σc). Then, as above, $\boldsymbol{A}$ contains a complete set of matrix units $\{E_{\lambda\mu}\}_{\lambda,\mu\in\Lambda}$ and consists of all operators $\sum_{\lambda,\mu}a_{\lambda\mu}E_{\lambda\mu}$ where $\sum_{\lambda,\mu}|a_{\lambda\mu}|^2<\infty$. Since for $A\in\boldsymbol{A}$, $\|A\|_\sigma^2=t(A^*A)=t(\psi(A)^*\psi(A)) =\|\psi(A)\|_\sigma^2$, and since any sum $\sum_{\lambda,\mu}a_{\lambda\mu}E_{\lambda\mu}\in\boldsymbol{A}$ is the limit in the Schmidt norm of finite sums, it is clear that if one sets $F_{\lambda\mu}=\psi(E_{\lambda\mu})$ for all λ, $\mu\in\Lambda$, then the $F_{\lambda\mu}$ form a complete set of matrix units for $\boldsymbol{B}$, and $\boldsymbol{B}$ consists exactly of all operators of the form $\sum_{\lambda,\mu}a_{\lambda\mu}F_{\lambda\mu} =\psi(\sum_{\lambda,\mu}a_{\lambda\mu}E_{\lambda\mu})$. Then $E_{\lambda\lambda}$ and $F_{\lambda\lambda}$ must be finite dimensional projections, and since $t(E_{\lambda\lambda})=t(F_{\lambda\lambda})$ for all $\lambda\in\Lambda$, all of the $E_{\lambda\lambda}$ and $F_{\lambda\lambda}$ must project onto subspaces of $\mathcal{H}$ of the same dimension. Let $E_{\lambda\lambda}(\mathcal{H})=\mathcal{H}_\lambda$ and $F_{\lambda\lambda}(\mathcal{H})=\mathcal{K}_\lambda$ for all λ, and let λ_0 be any fixed index in Λ. Choose some partial isometry W_{λ_0} with initial space $\mathcal{H}_{\lambda_0}$ and final space $\mathcal{K}_{\lambda_0}$. For each $\mu\in\Lambda$, $\mu\neq\lambda_0$, define $W_\mu=F_{\mu\lambda_0}W_{\lambda_0}E_{\lambda_0\mu}$. Easy calculations show that for all μ, W_μ is a partial isometry having initial

space $\mathcal{H}_\mu$ and final space $\mathcal{K}_\mu$, and also that $W_\mu E_{\mu\lambda_0} = F_{\mu\lambda_0} W_{\lambda_0}$. Let $W = \sum_{\mu\in\Lambda} W_\mu$. It is clear that this sum converges in the strong operator topology and that W is a partial isometry having initial space $\sum_{\mu\in\Lambda} \oplus \mathcal{H}_\mu$ and final space $\sum_{\mu\in\Lambda} \oplus \mathcal{K}_\mu$. Also $WE_{\mu\lambda_0} = F_{\mu\lambda_0} W$ for all $\mu\in\Lambda$. It follows that $WE_{\mu\lambda_0}W^* = F_{\mu\lambda 0}$ for all μ, and by taking adjoints we obtain $WE_{\lambda_0\mu}W^* = F_{\lambda_0\mu}$ for all μ. Since for any $\lambda, \mu\in\Lambda$, $F_{\lambda\mu} = F_{\lambda\lambda_0}F_{\lambda_0\mu}$, we have

$$(\pi) \qquad F_{\lambda\mu} = (WE_{\lambda\lambda_0}W^*)(WE_{\lambda_0\mu}W^*) = WE_{\lambda\mu}W^*$$

for all λ, $\mu\in\Lambda$. It is easy to see that $\mathfrak{M}(\boldsymbol{A}) = \mathcal{H} \ominus \sum_\mu \oplus \mathcal{H}_\mu$ and $\mathfrak{M}(\boldsymbol{B}) = \mathcal{H} \ominus \sum_\mu \oplus \mathcal{K}_\mu$. In addition, from the hypothesis we know that $\delta(\boldsymbol{A}) = \delta(\boldsymbol{B})$; it results that W can be extended to a unitary operator on $\mathcal{H}$ (which we continue to call W) that satisfies (π). Thus $W(\sum_{\lambda,\mu} a_{\lambda\mu}E_{\lambda\mu})W^* = \psi(\sum_{\lambda,\mu} a_{\lambda\mu}E_{\lambda\mu})$ for finite sums, and since both mappings preserve the Schmidt norm, it is clear that W implements ψ. This concludes the argument in the case that $\boldsymbol{A}$ is simple; in the general case, $\boldsymbol{A}$ can be decomposed as in Lemma 2.1, and this reduces the problem to the case already treated. We omit further details of that argument.

As indicated in the introduction, our plan is to use Theorem B to give a new proof of Theorem A, and the only remaining difficulty is disposed of in the following lemma.

LEMMA 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be any *-subalgebras of (σc) and let $\boldsymbol{A}$ and $\boldsymbol{B}$ be the closures in the Schmidt norm of $\mathcal{A}$ and $\mathcal{B}$, respectively. If ψ is a *-algebra isomorphism of $\mathcal{A}$ onto $\mathcal{B}$ such that $t(A_1A_2) = t(\psi(A_1A_2))$ for each pair of elements A_1 and A_2 of $\mathcal{A}$, then ψ can be extended to a *-algebra isomorphism of $\boldsymbol{A}$ onto $\boldsymbol{B}$ which is trace preserving on $\boldsymbol{A}\cap(\tau c)$.*

PROOF. The hypothesis guarantees that ψ preserves the Schmidt norm, and thus ψ can be extended to a *-algebra isomorphism ψ_0 of $\boldsymbol{A}$ onto $\boldsymbol{B}$ which preserves the Schmidt norm. Since (σc) is a Hilbert space under the inner product $[B, C] = t(C^*B)$, and ψ_0 can be regarded as a partial isometry which maps the subspace $\boldsymbol{A}$ of (σc) onto the subspace $\boldsymbol{B}$ of (σc), ψ_0 must also preserve inner products on $\boldsymbol{A}$. If $A\in\boldsymbol{A}\cap(\tau c)$, then by Lemma 2.1 there are elements B, $C\in\boldsymbol{A}$ such that $A = BC$. Thus $t(A) = t(BC) = [C, B^*] = [\psi_0(C), \psi_0(B^*)] = t(\psi_0(BC)) = t(\psi_0(A))$, and the proof of the lemma is complete.

PROOF OF THEOREM A. Let $\mathcal{A} = \{p(A, A^*)/p(x, y)\in P\}$ and $\mathcal{B} = \{p(B, B^*)/p(x, y)\in P\}$. Then $\mathcal{A}$ and $\mathcal{B}$ are *-subalgebras of (σc), and one can define a mapping $\psi: p(A, A^*)\to p(B, B^*)$ from $\mathcal{A}$ onto $\mathcal{B}$. To show that ψ is well defined and, in fact, a *-algebra isomorphism of $\mathcal{A}$ onto $\mathcal{B}$, it suffices to show that $p(A, A^*) = 0$ if and

only if $p(B, B^*)=0$. This is true because $p(A, A^*)=0$ if and only if $t(p(A, A^*)[p(A, A^*)]^*)=0$ and similarly for $p(B, B^*)$, and the hypothesis guarantees that

$$t(p(A, A^*)[p(A, A^*)]^*) = t(p(B, B^*)[p(B, B^*)]^*).$$

The proof is completed by applying Lemma 2.2 and Theorem B in that order, and observing that $\psi(A)=B$.

REMARK. With minor appropriate translations of terminology, the above proof of Theorem B becomes a very reasonable, direct proof of [6, Theorem 1] concerning the unitary equivalence of $n\times n$ complex matrices. (This result was improved by one of the authors in [2]).

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