STONE-WEIERSTRASS THEOREMS FOR THE STRICT TOPOLOGY

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1. Let X be a locally compact Hausdorff space, E a (real) locally convex, complete, linear topological space, and $\langle C^*(X, E), \beta\rangle$ the locally convex linear space of all bounded continuous functions on X to E topologized with the strict topology β. When E is the real numbers we denote $C^*(X, E)$ by $C^*(X)$ as usual. When E is not the real numbers, $C^*(X, E)$ is not in general an algebra, but it is a module under multiplication by functions in $C^*(X)$.

This paper considers a Stone-Weierstrass theorem for $\langle C^*(X), \beta\rangle$, a generalization of the Stone-Weierstrass theorem for $\langle C^*(X, E), \beta\rangle$, and some of the immediate consequences of these theorems. In the second case (when E is arbitrary) we replace the question of when a subalgebra generated by a subset S of $C^*(X)$ is strictly dense in $C^*(X)$ by the corresponding question for a submodule generated by a subset S of the $C^*(X)$-module $C^*(X, E)$. In what follows the symbols $C_0(X, E)$ and $C_{00}(X, E)$ will denote the subspaces of $C^*(X, E)$ consisting respectively of the set of all functions on X to E which vanish at infinity, and the set of all functions on X to E with compact support.

Recall that the strict topology is defined as follows [2]:

Definition. The strict topology (β) is the locally convex topology defined on $C^*(X, E)$ by the seminorms

$$\|f\|_{\phi,\nu} = \operatorname*{Sup}_{x\in X} |\phi(x)f(x)|_\nu$$

where ν ranges over the indexed topology on E and ϕ ranges over $C_0(X)$.

When E is the real numbers the above definition may be given simply as: a net $\{f_\alpha : \alpha\in A\}$ converges strictly to a function f iff $\{\phi f_\alpha : \alpha\in A\}$ converges uniformly to ϕf for each ϕ in $C_0(X)$. It is known that $C^*(X, E)$ is complete in the strict topology [2].

2. **Strict approximation in** $C^*(X)$. Stone-Weierstrass theorems were given for $C^*(X)$, under special restrictions by Buck [2]. The exact analogue of the classical theorem (for complex valued functions) was obtained by Glicksberg [3] as a corollary to a version of Bishop's generalized Stone-Weierstrass theorem [1]. The proof is not elementary. We give here a proof which follows quite simply from Buck's

Received by the editors May 1, 1964.

original theorems. These theorems were proved under the additional hypothesis that either (a) the subalgebra in question contained a function vanishing nowhere or (b) the space X on which the functions were defined was σ-compact.

THEOREM 1. *Let X be a locally compact Hausdorff space and let $\mathfrak{A}$ be a strictly closed subalgebra of $C^*(X)$ which separates points of X and which for each x in X, contains a function g with $g(x) \neq 0$, then $\mathfrak{A} = C^*(X)$.*

PROOF. Let $f \in C^*(X)$ and let $\phi \in C_0(X)$ and $\epsilon > 0$ be given. Let $K(\phi)$ be the σ-compact set outside of which ϕ vanishes identically. $K(\phi)$ may be replaced by a regularly σ-compact subset K of X (i.e. K is a countable union of compact subsets K_n, with K_n contained in the interior of K_{n+1}) which contains $K(\phi)$ and which is an open F_σ set [2]. K is a locally compact Hausdorff space in its relative topology and hence the strict topology is defined for $C^*(K)$, and ϕ is in $C_0(K)$. $\mathfrak{A}$ restricted to K is dense in $C^*(K)$ in the strict topology by (b) above, consequently there is a function g in $\mathfrak{A}$ such that $|\phi(x)f(x) - \phi(x)g(x)| < \epsilon$ for x in K. Since ϕf and ϕg vanish identically outside K, we have $\|g-f\|_\phi < \epsilon$. The conclusion now follows from the fact that $\langle C^*(X), \beta\rangle$ is complete.

The usual theorem for the complex case follows immediately from the above. We also note without proof the following obvious consequences of Theorem 1. (1) If X is a product of locally compact Hausdorff spaces which is locally compact, then every function in $C^*(X)$ may be strictly approximated by finite sums of finite products of functions of one variable on X. (2) There is a 1-1 correspondence between the β-closed ideals of $C^*(X)$ and the closed subsets of X. In particular there is a 1-1 correspondence between the maximal β-closed ideals and the points of X, and each β-closed ideal is the intersection of all β-closed maximal ideals which contain it.

Theorem 1 may be applied to obtain the following result.

THEOREM 2. *If X is a locally compact and σ-compact Hausdorff space, then $\langle C^*(X), \beta\rangle$ is separable iff X is metrizable.*

PROOF. Suppose that X is metrizable, hence since X is σ-compact, X is separable and satisfies the second axiom of countability. Let $\mathfrak{B}$ be a countable base for the topology on X. For each B in $\mathfrak{B}$, the complement of B, being closed, is the zero set of a function f_B in $C^*(X)$. Let $\mathfrak{A}$ be the countable algebra generated by $\{f_B | B \in \mathfrak{B}\}$ over the rationals. $\mathfrak{A}$ clearly satisfies the conditions of Theorem 1 and hence is β-dense in $C^*(X)$. Conversely let S be a countable β-dense subset of $C^*(X)$, and let $\mathfrak{A}_S$ be the countable algebra generated by S over the rationals. Let ϕ be any element of $C_0(X)$ which vanishes nowhere on X. Let $\phi\mathfrak{A}_S=\{\phi f | f\in\mathfrak{A}_S\}\subseteq C_0(X)$. By the classical Stone-Weierstrass theorem $\phi\mathfrak{A}_S$ is uniformly dense in $C_0(X)$. Let $\mathfrak{A}^*$ be the subset of $\phi\mathfrak{A}_S$ consisting of all functions f such that $\|f\|\leqq 1$. These functions may be used to define a function E on X into the unit cube $Q^w=\prod_f\{[0, 1]_f | f\in\mathfrak{A}^*\}$ by $E(x)_f=f(x)$. E is easily shown to be a homeomorphism, hence X is metrizable.

It is conjectured that separability for $\langle C^*(X), \beta\rangle$ under the sole hypothesis that X is locally compact and Hausdorff is characterized by the statement: $\langle C^*(X), \beta\rangle$ is separable iff X is metrizable and separable. Note that if X is metrizable and separable then X is necessarily σ-compact.

3. **Strict approximation in** $C^*(X, E)$. Let S be a subset of $C^*(X, E)$ and let x be an element of X, M a closed subspace of E. If $S(x)=\{\alpha\in E | \alpha=f(x)$ for some $f\in S\}$ is contained in M then we will say S is restricted to M at x. We will call a subset S of $C^*(X, E)$ unrestricted if for each maximal closed subspace M of E and each point x of X, there is a function f in S with $f(x)\notin M$.

THEOREM 3. *Let X be a locally compact Hausdorff space, and E a locally convex, complete, linear topological space. If S is an unrestricted submodule of $C^*(X, E)$, then S is strictly dense in $C^*(X, E)$.*

We first give a proof of the following fact due to Buck [2].

LEMMA. *$C_{00}(X, E)$ is strictly dense in $C^*(X, E)$.*

PROOF. Let $f\in C^*(X, E)$, $\phi\in C_0(X)$, a seminorm ν for E and $\epsilon>0$ be given. ϕf is in $C_0(X, E)$. Let $K\subseteq X$ be the compact set outside of which $|\phi(x)f(x)|_\nu<\epsilon/2$. There exists a function θ in $C_{00}(X)$ such that $\theta(x)=1$ for x in K and $\|\theta\|\leqq 1$. θf is in $C_{00}(X, E)$ and $|\phi\theta f(x)-\phi f(x)|_\nu<\epsilon$ for every x in X. Therefore f is in the strict closure of S.

PROOF OF THEOREM 3. In view of the preceding lemma we need only show that if S is an unrestricted submodule, then $C_{00}(X, E)$ lies within the strict closure of S. We show in fact that $C_{00}(X, E)$ is in the uniform closure of S (it is true that $C_0(X, E)$ is in the uniform closure of S).

Let f be any function in $C_{00}(X, E)$ and let a seminorm ρ for E and $\epsilon>0$ be given. Let $\Psi_\rho(x)=|f(x)|_\rho$. Ψ_ρ is in $C_{00}(X)$. Let K be the compact set outside of which Ψ_ρ vanishes identically. For each g in S, $\Psi_\rho g$ is in S and $\Psi_\rho g(x)=0$ for x not in K. Let x_0 be any point of K and suppose $f(x_0)=\xi$. By hypothesis there is a function g in S such that $g(x_0)=\xi$. Let $\Psi_\rho(x_0)=a$. If $a\neq 0$ let h_{x_0} be the function $a^{-1}\Psi_\rho g$. $h_{x_0}(x_0)$

$=\xi$, $h_{x_0}(x)=0$ for x not in K, and h_{x_0} is in S. Thus there exists a neighborhood U_{x_0} of x_0 such that $|h_{x_0}(y)-f(y)|_\rho<\epsilon$ for y in U_{x_0}. Suppose that $\Psi_\rho(x_0)=0$. Then let $h_{x_0}=\Psi_\rho g$. $|h_{x_0}(x_0)-f(x_0)|_\rho=|f(x_0)|_\rho=0$ and again by the continuity of seminorms we may choose a neighborhood U_{x_0} of x_0 with $|h_{x_0}(y)-f(y)|_\rho<\epsilon$ for y in U_{x_0}. In the usual manner we obtain a finite number of points x_i in K, of functions h_i in S, and neighborhoods U_i of x_i such that if y is in K then y is in U_i for some i and $|h_i(y)-f(y)|_\rho<\epsilon$. The set of neighborhoods U_i is a point finite open cover of K in its relative topology. Since K is completely regular and compact, there exist functions ϕ_i in $C^*(K)=C(K)$ such that $\phi_i(U_i)\subset[0, 1]$, ϕ_i vanishes identically in K outside $U_i\cap K$, and $\sum_i \phi_i(x)=1$ for each x in K. Since K is a compact subset of a completely regular space each ϕ_i may be extended to a function in $C^*(X)$, so that the functions $\phi_i h_i$ are defined for each i and are in S. The function $F=\sum_i \phi_i h_i$ is in S, is identically 0 outside K and $\|F-f\|_\rho<\epsilon$. Thus $C_{00}(X, E)$ lies in the uniform closure of S. From the preceding lemma it follows that S is strictly dense in $C^*(X, E)$.

COROLLARY 1. *Any subspace of $C^*(X, E)$ which is a $C_{00}(X)$ submodule is strictly dense in $C^*(X, E)$. A strictly closed $C_{00}(X)$ submodule is a $C^*(X)$ module.*

COROLLARY 2. *A function f in $C^*(X, E)$ is in the strict closure of the $C^*(X)$ submodule generated by an arbitrary subset S of $C^*(X, E)$ if and only if for any x in X and maximal closed subspace M of E, $S(x)\subseteq M$ implies $f(x)\in M$.*

Theorem 3 generalizes results obtained in [2]. We use the above results to characterize the strictly closed maximal submodules of $C^*(X, E)$.

If M is a maximal closed subspace of E and x is a fixed point of X, let $S_{x,M}=\{f\in C^*(X, E)\,|\,f(x)\in M\}$. Let y be any point of X with $y\neq x$, and $a\in E$, $b\in M$. Let ϕ_1 and ϕ_2 be functions in $C^*(X)$ such that $\phi_1(y)=1$, $\phi_1(x)=0$, and $\phi_2(y)=0$, $\phi_2(x)=1$. Let $h=a\phi_1+b\phi_2$. $h(x)=b$, $h(y)=a$ and h is in $S_{x,M}$. Thus $S_{x,M}(x)=M$ and $S_{x,M}(y)=E$ for each y in X with $y\neq x$. By Corollary 2, $S_{x,M}$ is a strictly closed submodule of $C^*(X, E)$. We show that it is a maximal submodule. Suppose that S' is a submodule of $C^*(X, E)$ such that $S'\supset S_{x,M}$, $S'\neq S_{x,M}$. Then there is a function f in S' such that $f(x)\notin M$. Let $f(x)=\lambda\in E$. E may be represented as a direct sum $M\oplus N$ where N is a complement of M. Then for each $\xi\in E$, $\xi=m+n$, with $m\in M$, $n\in N$, and so for some m_0 in M and n_0 in N, $\lambda=m_0+n_0$. There exists a function g in $S_{x,M}$ such that $g(x)=m_0$. The function $f'=f-g$ in S' has the property $f'(x)=n_0$.

Since M is of deficiency 1 in E each point of N is of the form αn_0, where α is a real number. Thus if ξ is any element of E, $\xi = m + \alpha n_0$, there is a function h in S', $h = g + \alpha f'$ (where g is a function in $S_{x,M}$) such that $h(x) = \xi$, hence $S'(x) = E$. Also $S'(y) = E$ for any point y in X. Now let j be any function in $C^*(X, E)$ and let $j(x) = \eta$. There is a function g_1 in S' with $g_1(x) = \eta$. Let $g_2 = j - g_1$. $g_2(x) = 0$ so g_2 is in $S_{x,M} \subset S'$. $j = g_1 + g_2$ so j is in S' thus S' is $C^*(X, E)$. Consequently $S_{x,M}$ is a strictly closed maximal submodule. Conversely it is easy to see that if S is a strictly closed maximal submodule of $C^*(X, E)$ then S is of the form $S_{x,M}$.

Note that unlike the situation for maximal ideals there may exist maximal submodules which are not strictly closed. Let E be any space (with the usual requirements) which admits a discontinuous linear functional L. Let N be the null space of L. N is a maximal subspace of E and is dense in E. Let x be any point of X and let S be the set of all functions f in $C^*(X, E)$ such that $f(x) \in N$. It is a routine verification that S is a $C^*(X)$ submodule of $C^*(X, E)$. We show that S is a maximal submodule. Let η be any element of E with η not in N and let f be a function in $C^*(X, E)$ with $f(x) = \eta$. If ξ is any point of E then $\xi = \alpha\eta + n$ with $n \in N$ since N is maximal. Let g be any function in $C^*(X, E)$ and let $g(x) = \lambda = \partial\eta + n$, $n \in N$. Let $g_1 = g - \partial f$. $g_1(x) = g(x) - \partial f(x) = n$ so g_1 is in S. $g = g_1 + \partial f$ so S is a maximal submodule. S is not strictly closed, however, by Corollary 2 of Theorem 3. We state the preceding facts as

THEOREM 4. *Let X be a locally compact Hausdorff space, E a locally convex, complete linear space. The strictly closed maximal submodules of $C^*(X, E)$ are the submodules of the form $S_{x,M}$. If E admits no discontinuous linear functional then every maximal submodule is closed.*

THEOREM 5. *Let X be a locally compact Hausdorff space and E a locally convex, complete linear space. A strictly closed submodule S of $C^*(X, E)$ is the intersection of all the maximal strictly closed submodules that contain it.*

PROOF. Let S be any strictly closed submodule of $C^*(X, E)$. Fix x in X and let M_x be the closure of $S(x)$ in E. Let $\mathfrak{M}(x, S)$ be the set of all maximal closed subspaces M of E which contain M_x. $M_x = \bigcap \{ M \mid M \in \mathfrak{M}(x, S) \}$ by the Hahn-Banach theorem for locally convex spaces. Clearly for each x in X, S is contained in $S_{x,M}$ when M is in $\mathfrak{M}(x, S)$. Let $S' = \bigcap \{ S_{x,M} \mid x \in X, M \in \mathfrak{M}(x, S) \}$. S' is a strictly closed submodule of $C^*(X, E)$ which contains S and by Corollary 2 of Theorem 3 each function f in S' is in S so that $S' = S$.

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WEAK LIMITS OF POWERS OF A CONTRACTION IN HILBERT SPACE[1]

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Let T be an operator, on the Hilbert space H, with $\|T\| \leqq 1$. Let

$$H_0 = \{x \mid \text{weak lim } T^n x = 0\}, \qquad H_1 = H_0^{\perp}.$$

We shall use the facts, proved in [1], that

1. $x \in H_0$ *if and only if* $\lim (T^n x, x) = 0$ (Theorem 3.1).
2. *On* H_1 *the operator* T *is unitary* (Theorem 1.1).

Given $x \in H$ let $x = x_0 + x_1$, where $x_0 \in H_0$ and $x_1 \in H_1$. The purpose of this note is to find conditions, on the sequence $(T^n x, x)$, that will imply that x_1 is generated by eigenvectors of T. This is related to the notion of mixing for ergodic transformations.

Theorem 1. *Let y be in the subspace generated by $T^n x$, $T^{*n} x$, $n = 1, 2, \cdots$. If* $\lim (T^n y, x) = 0$, *then* weak lim $T^n y = 0$.

Proof. Since H_0 and H_1 are invariant under T, it is enough to prove the theorem for the case when $x \in H_1$ (and thus also $y \in H_1$). If n_i is any subsequence of the integers and $z =$ weak lim $T^{n_i} y$, then z is orthogonal to $T^k x$, $k = 0, \pm 1, \pm 2, \cdots$ (since T is unitary on H_1). But z belongs to the subspace generated by $T^{\pm n} x$. Hence, $z = 0$ and, thus, weak lim $T^n y = 0$.

Corollary. *Let $P(\lambda)$ be a polynomial whose roots of modulus one are $\lambda_1, \cdots, \lambda_k$. If* $\lim (T^n P(T) x, x) = 0$, *then* $x = x_0 + x_1$ *where*: $x_0 \in H_0$, $x_1 = \sum_{i=1}^{k} z_i$, $T z_i = \lambda_i z_i$.

Proof. Let $x = x_0 + x_1$, where $x_0 \in H_0$ and $x_1 \in H_1$. By Theorem 1,

Received by the editors March 25, 1964.

[1] This work was partially supported by NSF Grants GP87 and G14736 at Northwestern University.