CONCERNING THE COMMUTATOR
SUBGROUP OF A RING

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This paper considers two independent results concerning \([A, A]\), the commutator subgroup of an associative ring \(A\), and generated by all elements \([a, b] = ab - ba\), where \(a\) and \(b\) are in \(A\). The first of these results sharpens those of [3], while the second uses the techniques of [6] to generalize [1] and [4]. These results are stated as

**Theorem 1.** Let \(A\) be a simple associative ring; then either \(A\) is a field or \([A, A]^2\), the subgroup generated by all products \(ab\) where \(a\) and \(b\) are in \([A, A]\), is \(A\).

**Theorem 2.** Let \(A\) be an associative ring such that \([A, A] = 0\), the subring generated by \([A, A]\), is \(A\) and let \(U\) be a Lie ideal of \([A, A]\), then either \([U, U], U\) = \(0\) or there exists a nontrivial (two-sided) ideal, \(R\), of \(A\) such that \(R \subseteq U^\perp\).

**Proof of Theorem 1.** Assume \(A\) is not \(4\)-dimensional over \(Z\), its center and a field of characteristic 2; if so, then a direct verification shows that \([A, A]^2 = A\). Let \(x, y \in [A, A]\) and \(a \in A\), then \([x, y]a = [x, ya] + y[a, x]\). Thus,

\[
(xy - yx)A \subseteq [A, A] + [A, A]^2 \quad \text{for all } x, y \in [A, A].
\]

Now for any \(b \in A\), \(b[x, y]a = [b, [x, y]a] + [x, y]ab\) and hence,

\[
A(xy - yx)A \subseteq [A, A] + [A, A]^2 \quad \text{for all } x, y \in [A, A].
\]

Therefore either (a) \([[A, A], [A, A]] = 0\), or (b) \(A = [A, A] + [A, A]^2\). (a) implies by [4] and [1] that \(A\) is a field, and (b) implies \([A, A]^2 = A\) by the use of the following lemma.

**Lemma 1 (Herstein).** Let \(A\) be a simple associative ring, neither a field nor \(4\)-dimensional over its center, \(Z\), a field of characteristic 2. Then \([A, A] \subseteq [A, A]^2\).

**Proof.** \([A, A]^2\) is obviously a Lie ideal of \(A\) and hence by [3] either is contained in \(Z\) or contains \([A, A]\). We now show that \([A, A]^2 \subseteq Z\) leads to a contradiction. Let \(a, b, c \in A\); then \(u = [a, b][a, c]\) and \(ua = [a, b][a, ca]\) are in \(Z\). Now if \(u \neq 0\), then the latter implies that \(a \in Z\) and hence \(u = 0\), which is false. Thus, for all \(a, b, c \in A\),

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\[ [a, b][a, c] = 0. \] An easy verification shows that this leads to \( a \in Z \), a contradiction. Thus the desired conclusion.

**Proof of Theorem 2.** We assume that \([A, A]^{-} = A\). To prove the theorem we need the following lemma.

**Lemma 2.** Let \( U \) be a Lie ideal of \([A, A]\). Then \( I = I(U) = \{ u \in U^{-} | \ u a \in U^{-} \text{ for all } a \in A \} \) is an ideal of \( A \) with the property that it contains every ideal of \( A \) which is a subset of \( U^{-} \).

**Proof.** The latter statement is obvious from the definition of \( I \).
It is also evident that \( I \) is a right ideal. Let \( b \in [A, A], a \in A, \) and \( u \in I \). Then, \( b(ua) - (ua)b \in U^{-} \), and \( bu - ub \in U^{-} \) which implies that \([A, A]I \subseteq I \). Thus, for all \( n \geq 1, [A, A]^{n}A, A \subseteq I \) and hence \( AI \subseteq I \). So, \( I \) is an ideal of \( A \). (The lemma also holds with \( U \) replacing \( U^{-} \) everywhere in the definition of \( I \).)

We are now in a position to prove Theorem 1. Suppose \([[U, U], U] \neq (0); \) then there exists \( x \in [U, U], y \in U \) such that \( xy - yx \neq 0 \). Since \([[U, U], A] \subseteq [U, [U, A]] \subseteq U, \) we have \([x, y] \in U \). Also, \([x, y]a = [x, ya] + y[x, a] \) for all \( a \in A \). By the previous remark, \([x, ya] \) and \([x, a]\) are in \( U \) and thus \([x, y]a \in U^{-} \) for all \( a \in A \). Thus, \( I \neq (0), \) and by Lemma 2, the theorem is proved.

This theorem can be strengthened to Theorem 3 for certain rings using an argument similar to [3] and the following lemma.

**Lemma 3 [5].** If a ring \( A \) has no nonzero right ideal, \( J, \) with \( a^{n} = 0 \) for all \( a \in J, \) \( n \) fixed, then \( A \) has a nonzero nilpotent (two-sided) ideal.

**Theorem 3.** Let \( A \) be a ring with no nilpotent ideals and such that \( 2x = 0 \) implies \( x = 0. \) Then either \( U^{-} \) contains a nontrivial ideal of \( A \) or \([U, U] \subseteq Z, \) the center of \( A \).

**Proof.** We have seen that \([x, y] \in I \) for all \( x \in [U, U], y \in U \). Thus, either \( U^{-} \) contains a nontrivial ideal of \( A \) or \([x, y] = 0 \) for all \( x \in [U, U], y \in U \). If the latter holds, then for all \( a \in A, \) \([x, [x, a]] = 0. \) Setting \( a = bc \) and expanding the resulting expression, we obtain \( 2[x, b][x, c] = 0 \) for all \( b, c \in A \) which yields, using the hypothesis,
\[
[x, b]^{2} = 0 \quad \text{for all } x \in [U, U], \ b \in A.
\]
Suppose \([x, a] = 0, x \in [U, U], \) and for all \( a \in [A, A]; \) then, since \([A, A]^{-} = A, x \in Z. \) Thus, assume that \( y = [x, b] \neq 0 \) for some \( b \in [A, A]. \) Then, \( y \in [U, U] \) and from (1) we have
\[
y^{2} = 0 \quad \text{and} \quad [y, d]^{2} = 0 \quad \text{for all } d \in A.
\]
Multiply (2) on the left by \( y \) and on the right by \( d \) and obtain \( (yd)^{2} = 0. \) Thus \( yA \) is a right ideal satisfying identity of Lemma 3. If \( yA \neq (0), \)
then we have a contradiction, while \( yA = (0) \) implies \( A \) being simple that \( y = 0 \), which also is a contradiction. Thus we have shown 
\[ [U, U] \subset Z. \]

This result indeed generalizes the work of [1] and [4].

**Theorem 4.** If \( A \) is simple (then \([A, A] = A \)) and \( U \) is a proper Lie ideal of \([A, A]\), then \( U \) is contained in the center of \( A \) except where \( A \) is of characteristic 2 and \( 4 \)-dimensional over \( Z \), a field of characteristic 2.

**Proof.** Define \([U, U] = U^{(1)} \) and \( U^{(n+1)} = [U^{(n)}, U^{(n)}] \) for all \( n \geq 1 \). Then, since \( A \) is simple, it has no nonzero nilpotent ideals. Thus, except in characteristic 2, \([U, U] \subset Z\) or \( U^2 = A \). If the former, then Theorems 7 and 9 of [4], in the case not characteristic 3, and Lemma 3 of [1] in this case implies \( U \subset Z \). Now, by these same results, if \( U^{(2)} \subset Z \), then \( U \subset Z \). Hence \([U^{(2)}] = A \). Thus, by Lemma 9 of [2] we have \([U^{(2)}, A] = [A, A] \), which contradicts \( U \) being proper. Lemma 1 of [1] yields the result when \( A \) is of characteristic 2.

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**References**

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