DIFFERENTIAL OPERATORS APPROXIMATELY OF THE
FORM \((1-(1/W(x))d/dx)^n\)

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1. Introduction. This paper studies the factorization of certain homogeneous linear differential operators \(L = \sum_{i=0}^{n} a_i(x) D^i\) into products \(b_0(x)(1-b_1(x)D) \cdots (1-b_n(x)D)\) of first-order factors, the functions \(a_i\) and \(b_i\) discussed in terms of asymptotic behavior as \(x\) tends to infinity in the complex plane. Among other results, we obtain information about the asymptotic behavior of the solutions of \(L(y) = 0\).

Speaking loosely, the operators \(P\) are assumed to be expressible in the form \(L = (1-W^{-1}D)^n + E_n(1-W^{-1}D)^{n-1} + \cdots + E_1(1-W^{-1}D) + E_0\) where each \(E_i \to 0\) as \(x \to \infty\), while \(\int_{a}^{b} W \to \infty\) as \(x \to \infty\). In the factorization \(L = b_0(1-b_1D) \cdots (1-b_nD)\) we obtain, it will be the case that \(b_0 \to 1\) and \(b_i/(W^{-1}) \to 1\) as \(i \to \infty\).

We use Strodt's asymptotic relations \(<, ~, \approx, \sim\), referring to a sort of sectorial neighborhood system of infinity in the complex plane. This system is denoted \(F(\alpha, \beta)\). (For an index of notation and terminology, see [2, Part IX, pp. 105–107].) Roughly, \(f < 1\) means \(f \to 0\); \(f \sim g\) means \(f/g \to 0\); \(f \sim g\) means \(f/g \to 1\), or \((f/g) - 1 < 1\); and \(f \approx g\) means \(f/g \sim c\), where \(c\) is a nonzero complex constant. Logarithmic monomials are functions of the form \(cx_0^m_1(\log x)^m_2(\log_2 x)^m_3 \cdots (\log_\tau x)^m_\tau\), with \(c\) complex and \(m_i\) real, \(\log_\tau\) being the \(\tau\)-fold iteration of \(\log\). Such functions are \(< 1\) and \(> 1\) according as the first nonzero \(m_i\) is negative or positive. A logarithmic domain is a set of functions which, together with all finite linear combinations thereof (with logarithmic monomials for coefficients), are either \(\sim\) logarithmic monomials or else are trivial (\(< \) every logarithmic monomial). The restriction, in Theorem I, to a logarithmic domain \(D\) has the effect of guaranteeing that the differences \(V-W, V \sim W\), will be \(<\)-comparable with \(W_0/\int W_0\) (which is \(\approx x^{-1}(\log x)^{-1} \cdots (\log_\tau x)^{-1}\) for some \(\tau\)).

The letters \(W, W_i, V, U\), etc. are used for functions which are \(\sim\) logarithmic monomials of the form

\[cx^{-1}(\log x)^{-1} \cdots (\log_{p-1} x)^{-1}(\log_p x)^{-1+\tau}(\log_{p+1} x)^{a_1} \cdots (\log_{p+s} x)^{a_s}, \quad \tau > 0,\]

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so that their $f_1^n$ diverges. (In fact, $\int f^1 \sim x^{-1} \log x \cdot \cdot \cdot (\log x^n) f(x)$ for any such $f$. See [3, Part V, Lemma $\xi$].) Logarithmic monomials of this sort form the divergence class. For every $M$ in this class and every $E<1$, $(M^{-1} E) E<1$; in particular, $DE<e^{-1} (\log x)^{-1} \cdot \cdot \cdot (\log x^q)^{-1}$ for $q = 0, 1, 2, \cdot \cdot \cdot$. This fact is used in Lemma 9.

An operator $L$ is unimajoral if $L(y) \sim 1$ whenever $y < 1$, while $L(y) \sim -1$ whenever $y > 1$; the $L$'s we study are unimajoral. It is known [2, §27] that whenever the coefficients $a_i(x)$ of a unimajoral $L = \sum_{i=0}^n a_i(x) D^i$ belong to a logarithmic domain and $a_n(x)$ is non-trivial, $L$ can be approximately factored: an approximate factorization sequence $W_1 \leq W_2 \leq \cdot \cdot \cdot \leq W_n$ of logarithmic monomials in the divergence class can be found such that $L = W_n \cdot \cdot \cdot W_1 + \sum_{i=1}^{n-1} E_i W_i \cdot \cdot \cdot W_1 + E_0$, where each $E_i < 1$ and by $W_i$ we mean the (unimajoral) operator $(1 - W_i D)$.

A similar expression, with different $E_i$'s (but always $< 1$), can be written using any sequence of functions $(V_1, V_2, \cdot \cdot \cdot, V_n)$ with $V_i \sim W_i$ in place of $(W_1, W_2, \cdot \cdot \cdot, W_n)$. We are interested in sequences $(V_1, V_2, \cdot \cdot \cdot, V_n)$ for which the corresponding $E_i$ are all 0 for $i < n$; such sequences yield exact factorizations: $L = (1 + E_n) V_n V_{n-1} \cdot \cdot \cdot V_1$. In [3, Part II], exact factorization is achieved for the case in which $(W_1, W_2, \cdot \cdot \cdot, W_n)$ is separated: $\exp(f(W_i - W_j)) < 1$ or $> 1$ for all pairs $i \neq j$. (For a precise definition, see [3, Part II].) This condition permits the determination of an exact factorization sequence $(V_1, V_2, \cdot \cdot \cdot, V_n)$ from certain quasi-linear algebraic differential equations treated in [2].

The present study treats unimajoral operators which possess approximate factorization sequences $W_1 = W_2 = \cdot \cdot \cdot = W_n: L = (W)^n + \sum_{i=0}^n E_i (W)^i$. The results of [3, Part II], which at first appear useless in case of such extreme lack of separation, are applied to an operator with a separated factorization sequence obtained from $L$ by some changes of variable. Theorem I, below, takes care of operators $L = (W)^n + \sum_{i=0}^n E_i (W)^i$ whose coefficients (which are in any case $< 1$) satisfy $E_i < (W)^{-n}$. We call such operators regular, this designation being suggested by analogy to the criterion for regular singular points in the classical treatment of linear differential equations.

Throughout this paper, integral signs should be construed as having a fixed lower limit, the upper limit serving as the independent variable. In situations involving $1/\sqrt{W}$, the domain of that variable will be subject to obvious and harmless restrictions.

2. Heuristics. Introduction of the new variable $s = -fW$ leads to $\dot{W} = 1 + d/ds$, so the operator $L = (W)^n + \sum_{i=0}^n E_i (W)^i$ becomes $(1 + d/ds)^n + \sum_{i=0}^n E_i (1 + d/ds)^i$. It is natural then to set $y = e^{-s}$ in
the equation \( L(y) = 0 \); this leads to \( ((d/ds)^n + \sum_{i=0}^{n} E_i (d/ds)^i) z = 0 \). In the trivial case where each \( E_i = 0 \), the substitutions \( s = e^t \) and \( z(s) = e^{-t} v(t) \) convert the equation \( (d/ds)^n z = 0 \) into
\[
e^{-(n+1)t} (-1)^n (1-n^{-1}d/dt) \cdots (1-1^{-1}d/dt) v = 0,
\]
in which we have an exactly factored operator with the separated factorization sequence \((1, 2, \ldots, n)\). The solutions \( v(t) \) are, of course, \( \approx \exp f_1, \exp f_2, \ldots, \exp f_n \). In case the \( E_i \) are not all zero, the same substitution in the equation \( ((d/ds)^n + \sum_{i=0}^{n} E_i (d/ds)^i) z = 0 \) leads (by straightforward calculation) to a unimajoral equation for which \((1, 2, \ldots, n)\) is an approximate factorization sequence, provided the \( E_i \) satisfy the condition \( E_{i} \leq W^{-(n-i)} \) \((i = 0, 1, \ldots, n)\). Since \((1, 2, \ldots, n)\) is separated, we may apply Theorem II of [3] to obtain an exact factorization \( V_n V_{n-1} \cdots V_1 v(t) = 0 \) where \( V_1 \sim 1, V_2 \sim 2, \ldots, V_n \sim n \), and solutions \( v(t) \) which are \( \approx \exp \int V_i, i = 1, 2, \ldots, n \).

3. **Definition.** If \( L = (W)^n + \sum_{i=0}^{n} E_i (W)^i \) with each \( E_i \leq 1 \) and \( W_0 \) in the divergence class. Then either there exists no \( W \) in \( D \) such that \( L \) is regular with respect to \( W \), or else \{ \( W: W \in D \) and \( L \) is regular with respect to \( W \) \} is precisely one of the equivalence classes in \( D \cap \{ W: W \sim W_0 \} \) defined by the equivalence relation \( \equiv \) where \( U \equiv V \) means \( U - V \leq W_0/\int W_0 \). If \( L \) is regular with respect to \( W \), then \( L \) can be expressed as a product of first-order factors: \( L = uV_n \cdots V_1 \), where \( V_i \equiv W \) for each \( i \). Conversely, if \( L \) is expressible as such a product, then \( L \) is regular with respect to \( W \). If \( L \) is regular with respect to \( W \), the solutions of \( L(y) = 0 \) are generated by a basis consisting of functions \( y_0, y_1, \ldots, y_{n-1} \) where \( y_0 < y_1 < \cdots < y_{n-1} \). For each \( i, y_i \sim \exp \int W_i \) for a certain \( W_i \sim W \), and \( \log(y_i/y_{i+1}) \sim -\log \int W \).

**Proof.** The theorem summarizes the following lemmas.

5. **Lemma.** Let \( L \) be regular with respect to \( W \). Then there exist functions \( W_0, W_1, \ldots, W_{n-1} \) satisfying \( W_i - W \sim W/fW \) for \( i > 0 \), \( W_0 - W \leq W/fW \), such that the equation \( L(y) = 0 \) has solutions \( y_0, y_1, \ldots, y_{n-1} \) satisfying \( y_i \sim \exp \int W_i, i = 0, 1, \ldots, n-1 \).

**Proof.** Let \( f = s^n e^{-s} \), where \( s(x) = -f^2 W \). We have
\[
\bar{W}(f) = ns^{n-1} e^{-s}, (W)^2(f) = n(n-1)s^{n-2} e^{-s}, \ldots, (W)^n(f) = n! e^{-s}.
\]
Using the relation $W(Zz) = (W(Z))(M(z))$, we obtain

\[ W(fz) = ns^{n-1}e^{-s}Q_1(z), \quad (W)^2(fz) = n(n-1)s^{n-2}e^{-s}Q_2Q_1(z), \ldots, \quad (W)^n(fz) = n!e^{-s}Q_n \cdots Q_1(z) \]

where $Q_i = -(n-i+1)W/iW$ for each $i$. Hence $L(fz)$ may be written $n!e^{-s}L'(z)$ where $L' = \hat{Q}_n \cdots \hat{Q}_1 + \sum_{i=0}^{n-1} (E_i s^n/(n-i)! \hat{Q}_i \cdots \hat{Q}_1$. 

(All this is verified in an elementary way.) Since $L$ is regular with respect to $W$, all but the leading coefficient in $L'$ are $< 1$. $L'$ is thus a unimajoral operator; furthermore, its factorization sequence $(Q_1, \ldots, Q_n)$ is separated. It follows from [3, Theorem I] that there exists a representation of $L'$ in the form $v\hat{U}_n \cdots \hat{U}_1$ with $v \sim 1$ and $U_i \sim Q_i$ for each $i$. According to [3, Theorem II], the equation $L'(z) = 0$ has solutions $z_i$ with $z_i \sim \exp \int U_i$, $j = 1, \ldots, n$. The corresponding solutions of $L(y) = 0$ are of the form $e^{-s}y_1$, which may be expressed as the product of a function $= 1$ and $\exp \int (W + U_j + nW/W)$.

6. **Lemma.** In Lemma 5 we have $y_0 \sim y_1 \sim \cdots \sim y_{n-1}$.

**Proof.** It is clear that $y_i/y_{i+1} \sim \exp \int (W_i - W_{i+1})$ and that $W_i - W_{i+1} \sim -W/W$. Using [3, Part V, Lemma 3], we see that the latter is $\sim -\tau x^{-\eta} \log x^{-1}$, for a positive real $\tau$ and an integer $p \geq 0$. Our conclusion then follows from [3, Part V, Lemma 3].

7. **Lemma.** In Lemma 5 we have $\log(y_i/y_{i+1}) \sim -\log fW$.

**Proof.** The relation $y_i/y_{i+1} \sim \exp \int (W_i - W_{i+1})$ immediately implies

\[ y_i/y_{i+1} = (1 + E) \exp \int (W_i - W_{i+1}) \]

where $E \sim 1$. Thus

\[ \log(y_i/y_{i+1}) = \int (W_i - W_{i+1}) + \log(1 + E). \]

Since $W_i - W_{i+1} \sim -W/W$, $\int (W_i - W_{i+1}) \sim -\log fW$ (this is seen with the help of [3, Part V, Lemma 3]), which is $> 1$. The term $\log(1 + E)$ is seen to be $< 1$, using [1, §27]. Hence we have $\log(y_i/y_{i+1}) \sim -\log fW$.

8. **Lemma.** If $L$ is regular with respect to $W$, there exist functions $V_1, V_2, \ldots, V_n$ such that $V_1 \sim V_2 \sim \cdots \sim V_n \sim W$, $V_i \sim W/W$ for each $i$, and $L = uV_1 \cdots V_n$ with $u \sim 1$. 

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Proof. As in the proof of Lemma 5, we have \( L(y) = n!e^{-\lambda} \hat{U}_n \cdot \cdot \cdot \hat{U}_1(y/f) \). Writing \( \hat{U}_1 = -(1+h_1)W/W \) with \( h_1 \ll 1 \), we have \( \hat{U}_1(y/f) = (\hat{U}_1(1/f) \hat{V}_1(y) \) where \( \hat{V}_1 = U_1(\hat{U}_1(1/f)) \) and \( \hat{U}_1(1/f) = \hat{U}_1(s^{-n}e^s) \)
\[
= s^{-n}e^s + s(n(1+h_1)Ds)^{-1}(-ns^{-n+1}e^{Ds} + s^{-n}e^sDs) 
= f^{-1}(1+h_1)^{-1}(h_1+s/n).
\]
Thus \( V_1 = -(nDs/s)(1+h_1)f^{-1}(1+h_1)^{-1}(h_1+s/n)f = -Ds(h_1n/s+1) = W(1-h_1n/fW) \). From this we see that \( V_1 - W < W/fW \).

Now \( \hat{U}_1(1/f) = ws^{-n}e^s \) where \( w \sim 1/n \). Operating on \( (\hat{U}_1(1/f)) \hat{V}_1 \) with \( \hat{U}_2 \), we get \( (\hat{U}_2(w)) \hat{U}_2(s^{-n}e^s \hat{V}_1) \) where \( \hat{U}_2 \sim U_2 \sim -(n-1)Ds/s \). Writing \( \hat{U}_2 = -(n-1)(Ds/s)(1+h_2) \) with \( h_2 \ll 1 \), and proceeding exactly as above, we obtain \( \hat{U}_2 \hat{U}_1(y/f) = ws^{-n}e^s \hat{V}_2 \hat{V}_1(y) \) where \( w' \sim (n(n-1))^{-1} \) and \( V_2 = W(1-h_2(n-1)/fW) \). Continuing in this way, we eventually transform \( \hat{U}_n \cdot \cdot \cdot \hat{U}_1(y/f) \) into \( w''e^s \hat{V}_n \cdot \cdot \cdot \hat{V}_1(y) \), where \( w'' \sim 1/n! \) and \( V_i - W < W/fW \) for each \( i \). This gives us \( L = u\hat{V}_n \cdot \cdot \cdot \hat{V}_1 \), concluding the proof.

9. Lemma. If \( V \sim W \) and \( V - W < W/fW \) (equivalently: \( V - W \sim V/fV \)), then \( L \) is regular with respect to \( W \) iff \( L \) is regular with respect to \( V \).

Proof. Assume \( V - W < W/fW \). Then
\[
W = (V/W)V + (1 - V/W) = (1 + g)V - g
\]
where \( g = (V - W)/W \ll 1/fV \). Thus \( \hat{W} \) is regular with respect to \( V \).

One proves by induction that the operators \( (\hat{W})^p \) \( (p = 2, 3, \cdot \cdot \cdot) \) are regular with respect to \( V \): If true for \( p = k - 1 \), we have \( (\hat{W})^{k-1} = (\hat{V})^{k-1} + \sum_{i=0}^{k-1} F_i(\hat{V})^i \) with \( F_i < (\hat{V})^{i-k} \) and
\[
(\hat{W})^p = ((1+g) \hat{V} - g) \left( (\hat{V})^{k-1} + \sum_{i=0}^{k-1} F_i(\hat{V})^i \right).
\]
That the latter is regular with respect to \( V \) follows from the fact that \( \hat{V}(F_i(\hat{V})^i)(y) = F_i(\hat{V})^{i+1}(y) - (DF_i/V)(\hat{V})^i(y) \) where \( F_i < (\hat{V})^{i+1-k} \) by the induction hypothesis, and \( DF_i/V < (\hat{V})^{i-k} \). The latter inequality is seen by writing \( F_i = h(\hat{V})^{i+1-k} \) with \( h \ll 1 \) and noting that \( Dh/V < (\hat{V})^{i-k} \) (because \( Dh < x^{-1}(log x)^{-1} \cdot \cdot \cdot (log x)^{-1} \) if \( h \ll 1 \)). The regularity of \( L \) with respect to \( V \) follows readily from the regularity of the \( (\hat{W})^p \) with respect to \( V \) and the assumption that \( L \) is regular with respect to \( W \). The converse follows from the symmetry of \( V \) and \( W \) in this discussion.
10. **Lemma.** If \( L = u \tilde{V}_n \cdots \tilde{V}_1 \), where \( u \sim 1 \), \( V_1 \sim V_2 \sim \cdots \sim V_n \sim W \) and \( V_i - W < W/\int W \) for each \( i \), then \( L \) is regular with respect to \( W \).

**Proof.** With the operators \( \tilde{V}_i \) expressed in the form \( (1 + g_i) \tilde{W} - g_i \), where \( g_i = 1 - W/V_i < 1/\int W \), the proof of this lemma closely parallels the proof of Lemma 9.

11. **Lemma.** If \( L \) is regular with respect to \( W \), and \( V \sim W \) but \( V < W \), then \( L \) is not regular with respect to \( V \).

**Proof.** Suppose \( L \) is regular with respect to \( V \). As a consequence of Lemmas 5 and 6, there exist solutions \( y_0 < y_1 < \cdots < y_{n-1} \) with \( y_i \sim \exp \int W \) and \( W_i - W \sim i W/\int W \) for \( i > 0 \), while \( W_0 - W < W/\int W \). At the same time we have solutions \( y_0 < y_1 < \cdots < y_{n-1} \) with \( y_i \sim \exp \int V \), \( V_i - V \sim i V/\int V \) for \( i > 0 \), while \( V_0 - V < V/\int V \). Since the \( y_i \) and the \( y_i \) constitute two bases for the space of solutions of \( L(y) = 0 \), it follows readily that \( y_0 = \tilde{y}_0 \), whence \( y_0/\tilde{y}_0 \sim 1 \) and \( \log(y_0/\tilde{y}_0) < 1 \). But \( y_0/\tilde{y}_0 \sim \exp \int(W_0 - V_0) \), and \( W_0 - V_0 = (W_0 - W) + (W - V) + (V - V_0) \sim W/\int W \). From this it follows that \( \log(y_0/\tilde{y}_0) \sim \int(W_0 - V_0) > 1 \), a contradiction. Therefore \( L \) is not regular with respect to \( V \).

12. **Remark.** There exist unimajoral operators \( L \) of the form \((\tilde{W})^n + \sum_{i=0}^{n} E_i(\tilde{W})^i \) such that \( L \) is not regular with respect to \( V \) for any \( V \).

For example, \((1 - (1+1/x)^{-1}D)^2 (1 - D)\) can be expressed in the form \((\tilde{W})^2 + E_1 \tilde{W} + E_0 \), with \( E_0, E_1 < 1 \), for \( W = 1 \) (indeed, for any \( W \sim 1 \)). That this \( L \) is not regular with respect to \( V \) for any \( V \sim 1 \) may be seen as follows: Suppose \( L \) is regular with respect to \( V \sim 1 \). Then by Lemma 5 there exist \( W_0 \) and \( W_1 \) with \( W_0 - V < V/\int V \) and \( W_1 - V \sim V/\int V \) such that the solutions of \( L(y) = 0 \) are generated by \( y_0 \sim \exp \int W_0 \) and \( y_1 \sim \exp \int W_1 \). In this case \( W_1 - W_0 \sim V/\int V \) and \( V/\int V \sim 1/x \), so \( y_1/y_0 \sim e^x \) where \( f \sim \log x \) and \( \log(y_1/y_0) \sim \log x \). But one sees directly that the solutions of \( L(y) = 0 \) are generated by \( e^x \) and \( x^2 e^x \), so \( y_1/y_0 \) must be \( x^2 \), whence \( \log(y_1/y_0) \sim 2 \log x \), a contradiction.

**References**

