CHARACTERIZATION OF TOTALLY UNIMODULAR MATRICES

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In this paper A will always denote a matrix with entries equal to 1, -1 or 0. A is *totally unimodular* if every square submatrix has a determinant equal to 1, -1 or 0. A submatrix A_J^I of A is said to be *Eulerian* [1] if

$$(\forall_I i): \sum_{j\in J} A_j^i \equiv 0 \bmod 2$$

and

$$(\forall_J j): \sum_{i\in I} A_j^i \equiv 0 \bmod 2.$$

We published in [2] and also in [6] a proof of:

THEOREM 1. *A is totally unimodular if and only if every square Eulerian submatrix is singular.*

We exposed the proof or the sufficient condition of this theorem as well as the one of Statement 3 at the Seminary on Combinatorial Problems of the Faculté des Sciences de Paris in January 1962. A. Ghouila-Houri gave independently a characterization [3] which could have been deduced however from our results.

R. Gomory gave us recently a new proof of the sufficient condition. His proof, combined with ours, allows us to prove a simpler characterization:

THEOREM 2. *A is totally unimodular if and only if for every (square) Eulerian submatrix A_J^I: $\sum_{i\in I;\, j\in J} A_j^i \equiv 0 \bmod 4$.*

The following Statements 1 and 2 clearly prove the sufficient condition of Theorem 1.

I. STATEMENT 1 (R. GOMORY). *If there exists a submatrix B of order n in the matrix A with $|\mathrm{Det}(B)| > 2$, there exists a square submatrix Q of B with $|\mathrm{Det}(Q)| = 2$.*

PROOF. Let $D = [BI]$, where I is the unit matrix, and let $\mathcal{C}$ be the class of all matrices obtained by unimodular row transformations of D, with the property: $(\forall_{\mathcal{C}} C)(\forall j)(\forall i): C_j^i \in \{1, -1, 0\}$ and C contains n different unit column vectors.

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Let F be a matrix in $\mathcal{C}$ with the greatest number of unit vectors among its first n columns. At least one vector, F_k, $k \leqq n$ is not a unit vector. For there cannot be in $\mathcal{C}$ a matrix of the form $[IG]$, since $|\mathrm{Det}(B)| > 2$.

Let us prove that among all possible choices of $n-1$ unit vectors such that $F_{J(r)} = [F_k, e_{r_1}, \cdots, e_{r_{n-1}}]$ contains the set F_j, $j \in T$ of unit vectors of the first n columns, at least one is such that $F_{J(r)}$ is unimodular.

If this was not true, $F_{J(r)}$ would be singular for each of those choices and to each of those choices would correspond a null coordinate of F_k.

Let F_k^i, $i \in S$ be the set of those null coordinates. Thus $F_{T\cup\{k\}}^S$ would be a null matrix $(n-|T|)\times(|T|+1)$ and consequently the matrix defined by the first n columns of F would be singular, contrary to the hypothesis that $\mathrm{Det}(B) > 2$.

Let e_1 be the vector which is not in the set $e_{r_1}, \cdots, e_{r_{n-1}}$; $F_{J(r)}$ (which will be denoted F_J for simplicity) being unimodular. $F_J^{-1}F$ cannot belong to $\mathcal{C}$, because it has one more unit vector than F in its first n columns. However, $F_J^{-1}F$ contains a unit matrix, thus one of the entries of $F_J^{-1}F$ is not 1, -1 or 0. Let us point out that (after rearrangement of rows and columns),

$$F_J = \begin{vmatrix} \epsilon_1 & 0 & \cdots & 0 \\ \epsilon_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \epsilon_n & 0 & \cdots & 1 \end{vmatrix},$$

where $\epsilon_1 \in \{1, -1\}$, and

$$F_J^{-1} = \begin{vmatrix} \epsilon_1 & 0 & \cdots & 0 \\ -\epsilon_2\epsilon_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\epsilon_n\epsilon_1 & 0 & \cdots & 1 \end{vmatrix}.$$

F_J^{-1} is the product of elementary matrices, $T_p, \cdots, T_1$, where p is the number of nonzero ϵ_i. Concretely,

$$T_1 = \begin{vmatrix} \epsilon_1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix},$$

$$T_p = \begin{vmatrix} 1 & 0 \cdots 0 \\ 0 & 1 \cdots 0 \\ \vdots & \\ 0 & \ddots \\ -\epsilon_s & \vdots \;\ddots \\ 0 & \\ \vdots & \\ 0 & 0 \cdots 1 \end{vmatrix}.$$

Let q be the smallest integer such that one of the entries of $F^* = T_q T_{q-1} \cdots T_1 F$ is not 1, -1 or 0.

Clearly, T_q is the transformation which adds (subtracts) the first row of $T_{q-1} \cdots T_1 F$ to (from) the ith row of this matrix. Then, the entries of F^* which are not 1, -1 or 0 are necessarily ± 2 and they must all be in the ith row. Let F_j^{*i} be one of those entries; necessarily, $j \notin J$. Moreover, $T_q T_{q-1} \cdots T_1 F_k$ has one more null coordinate than $T_{q-1} \cdots T_1 F_k$ which is precisely F_k^{*i}. Suppressing e_i of $F^*_{J \cup \{j\}}$, one finds $F^*_{J'}$, which has $n-2$ unit column vectors. The submatrix $F^{*\{1\} \cup \{i\}}_{\{k\} \cup \{j\}}$ is necessarily

$$\begin{vmatrix} \pm 1 & \pm 1 \\ 0 & \pm 2 \end{vmatrix},$$

thus $\operatorname{Det}(F^*_{J'}) = \pm 2$. But $\operatorname{Det}(F^*_{J'}) = \operatorname{Det}(D_{J'})$, since $F^*_{J'}$ is obtained from $D_{J'}$ by a unimodular transformation. As $D = [BI]$, $\operatorname{Det}(D_{J'})$ is equal to the determinant of one of the square submatrices of B.

STATEMENT 2 (R. GOMORY). *If for every Eulerian square submatrix* $A_{J'}^{I'}$ *of* A, $\operatorname{Det}(A_{J'}^{I'}) = 0$, *then, for every square submatrix* A_J^I *of* A, $\operatorname{Det}(A_J^I) \equiv 0 \bmod 2$ *implies* $\operatorname{Det}(A_J^I) = 0$.

PROOF. Let $\mathcal{C}$ be the class of all square submatrices A_J^I with $\operatorname{Det}(A_J^I) = 2k$, k nonzero. Let us suppose that $\mathcal{C} \neq \varnothing$. Let B be any matrix of minimum order. There exists a vector x with integral coordinates which are not all zero or even, such that $Bx \equiv 0 \bmod 2$, since $\operatorname{Det}(B) \equiv 0 \bmod 2$. Let B_K be the set of column vectors of B whose coefficients are the odd coordinates of x. These vectors are linearly dependant modulo 2, thus the determinants of the square submatrices B_K^I of B_K are null modulo 2.

As they cannot be all zero, since $\operatorname{Det}(B) \neq 0$, one of them must be even and nonzero.

Since there cannot exist a square matrix B_K^I in $\mathcal{C}$ which would be a proper submatrix of B, K is necessarily the set of all column vectors of B. Thus the row sums of B are even. The same argument applied to B^T proves that B is Eulerian, hence singular, which implies that contrary to definition, $\mathfrak{C}$ would contain a singular matrix.

II. We proved in [2] and also in [6] the following statement.

STATEMENT 3. *If A is totally unimodular, then to each vector x, with $Ax \equiv 0 \bmod 2$, corresponds a vector y whose coordinates are* 1, -1 *or* 0, *with $Ay=0$ and $y \equiv x \bmod 2$.*

With the help of Statement 3, we shall now prove the necessary condition of Theorem 2. It will be used for proving Statement 4 which with the previous statements will be used to prove the sufficient condition of Theorem 2.

PROOF OF THE NECESSARY CONDITION OF THEOREM 2. For every Eulerian submatrix A_J^I of the totally unimodular matrix A, we have

$$(1) \qquad (\forall_I i): \sum_{j \in J} A_j^i x_j \equiv 0 \bmod 2;$$

where $x_j=1$ for each j in J. Thus, by Statement 3, there exists a vector y with coordinates 1 or -1, for which

$$(2) \qquad (\forall_I i): \sum_{j \in J} A_j^i y_j = 0.$$

If y has at least one negative coordinate, let w be the vector y where the negative coordinates have been replaced by 1.

As A_J^I has an even number of nonzero entries in each column,

$$(3) \qquad \sum_{i \in I; j \in J} A_j^i y_j \equiv \sum_{i \in I; j \in J} A_j^i w_j \bmod 4.$$

Then, by (2),

$$(4) \qquad \sum_{i \in I; j \in J} A_j^i \equiv 0 \bmod 4.$$

STATEMENT 4. *Let A_J^I be a square Eulerian submatrix of a matrix A, such that every proper submatrix of A_J^I is totally unimodular, then* $\sum_{i \in I; j \in J} A_j^i \equiv \operatorname{Det}(A_J^I) \bmod 4$.

In the case where A_J^I is singular, the necessary condition of Theorem 2 proves Statement 4. So, let A_J^I be nonsingular. As

$$(5) \qquad \sum_{j \in J} A_J^{I-\{k\}} x_j \equiv 0 \bmod 2,$$

where $x_j=1$, for all j in J, there exists a vector y with coordinates 1 or -1 (Statement 3), for which

$$\sum_{j\in J} A_J^{I-\{k\}} y_j = 0. \tag{6}$$

Let y^D be the diagonal matrix where the jth diagonal element is y_j and let α be the number: $\sum_{j\in J} A_j^k y_j$; finally, let $B_J^I = A_J^I y^D$. Then, by (6),

$$\sum_{i\in I;\, j\in J} B_j^i = \alpha. \tag{7}$$

Since every column of B_J^I has an even number of nonzero elements, by the same argument as for the proof of the necessary condition of Theorem 2,

$$\sum_{i\in I;\, j\in J} A_j^i \equiv \alpha \mod 4. \tag{8}$$

It suffices now to prove that

$$\mathrm{Det}(A_J^I) \equiv \alpha \mod 4. \tag{9}$$

Let v be the column vector whose coordinates are zero, except the kth which is α. Then

$$y = (A_J^I)^{-1} v. \tag{10}$$

Each element of the kth column of $(A_J^I)^{-1}$ must be $1/|\alpha|$ since for all j, $|y_j| = 1$. But each entry of the adjoint of A_J^I is 1, -1 or 0. Thus $|\mathrm{Det}(A_J^I)| = |\alpha|$, which proves (9).

III. Proof of the sufficient condition of Theorem 2. We shall prove that if for every square Eulerian submatrix A_J^I of A, $\sum_{i\in I, j\in J} A_J^I \equiv 0 \mod 4$, then for those matrices $\mathrm{Det}(A)_J^I = 0$. Theorem 1 will end the proof.

Let $\mathcal{C}$ be the class of Eulerian square submatrices A_J^I with $\mathrm{Det}(A_J^I) \neq 0$. Assume $\mathcal{C}$ is not empty. Let B be any matrix of $\mathcal{C}$ of minimum order. Then for every square Eulerian proper submatrix B_J^I of B, $\mathrm{Det}(B_J^I) = 0$. By Theorem 1, this proves that every proper submatrix of B is totally unimodular, and Statement 4 then proves that $\mathrm{Det}(B) \equiv 0 \mod 4$. On the other hand, applying Statement 1 to B, one sees that $|\mathrm{Det}\, B| \leq 2$. Then $\mathrm{Det}(B) = 0$, and contrary to the hypothesis, $\mathcal{C}$ would contain a singular matrix.

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PRIME RINGS WITH MAXIMAL ANNIHILATOR AND MAXIMAL COMPLEMENT RIGHT IDEALS[1]

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1. **Introduction.** Let R be a prime ring with a maximal annihilator right ideal and a maximal complement right ideal. Then there is a division ring D such that either R is isomorphic to a right order in the complete ring of linear transformations of a finite dimensional D-space, or for each positive integer n there is a subring $R^{(n)}$ of R which is isomorphic to a right order in the complete ring of linear transformations of an n-dimensional D-space. This is related to a result of N. Jacobson [2, p. 33] and extends a theorem of A. W. Goldie [1; Theorem 4.4] that a prime ring with maximum conditions on annihilator right ideals and complement right ideals is a right order in a simple ring with minimum condition on right ideals. R is also isomorphic to a weakly transitive ring of linear transformations of a vector space. This is a generalization of a theorem of R. E. Johnson [4; 3.3].

2. We assume throughout that R is a prime ring. The notation R_r^Δ (R_l^Δ) is used to denote the right (left) *singular ideal* of R, and L_r^* (L_l^*) is the lattice of closed right (left) ideals of R. An R-module is *uniform* if each pair of nonzero submodules has nonzero intersection. A right (left) ideal of R is *uniform* if it is uniform as right (left) R-module. For other definitions and notation see [6].

THEOREM 1. *R contains a maximal annihilator right ideal and a maximal complement right ideal if and only if $R_r^\Delta = (0)$ and L_r^* is atomic.*

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