ment for A, and since B is simple, A is also a direct sum of simple algebras.

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SOME PROPERTIES OF LOCALIZATION AND NORMALIZATION

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In a recent note [**1**], S. Abhyankar has given some lemmas concerning localization and normalization for noetherian rings without nilpotent elements. We give a characterization of those rings in which every prime ideal is maximal (Proposition 1) and deduce generalizations of Abhyankar's results (cf. Corollary 1 and Corollary 2).

Preliminaries. A *ring* will always be a nonnull commutative ring with identity.

For properties of rings of quotients see [**3**, §§9–11 of Chapter IV]. Recall that if R is a ring with total quotient ring K, and if M is a multiplicative system in R, then we may identify the ring of quotients R_M with a subring of K_M. When this is done, the total quotient ring of K_M is also the total quotient ring of R_M.

Denote by g_M the canonical map of K into K_M; the restriction of this map to R is then the canonical map of R into R_M; the restricted map may also be denoted by g_M without fear of confusion. If M consists of all the powers of a single element f, then we write R_f, K_f, g_f, in place of R_M, K_M, g_M.

If Q is a minimal prime ideal in R, and M is the complement of Q in R, then QR_M, being the only prime ideal in R_M, consists entirely of zerodivisors (in fact, of nilpotents). Consequently, if $x \in Q$, then $g_M(x)$ is a zerodivisor, and it follows easily that x is a zerodivisor. Thus *any minimal prime ideal in a ring consists entirely of zerodivisors.*

Proposition 1. *For a ring R, the following statements are equivalent*:

(1) *Every prime ideal in R is maximal.*

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(2) *For every multiplicative system M in R, the canonical map $g_M: R \to R_M$ is surjective.*

(3) *Any homomorphic image of R is its own total quotient ring.*

PROOF. (1)→(3). If $\bar{R}$ is a homomorphic image of R, then every prime ideal in $\bar{R}$ is maximal; therefore every prime ideal in $\bar{R}$ is minimal and so consists entirely of zerodivisors. It follows that every non-unit in $\bar{R}$ is a zerodivisor; (3) is thereby proved.

(3)→(2) is clear, since R_M is contained in the total quotient ring of $g_M(R)$.

(2)→(1), for if $P<Q$ are proper prime ideals, ($<$ means "*strictly* contained in") and if f is any element in $Q-P$, then P contains the kernel of $g_f: R \to R_f$; *a fortiori*, Q contains the kernel, whence $g_f(Q)$ is a proper ideal in $g_f(R)$. But $g_f(f) \in g_f(Q)$ is a unit in R_f; hence $g_f(R) < R_f$.

COROLLARY 1. *For a ring R with total quotient ring K, the following statements are equivalent:*

(1) *Every prime ideal in R consisting entirely of zerodivisors is a minimal prime ideal.*

(2) *For every multiplicative system M in R, $g_M(K)$ is the total quotient ring of R_M.*

PROOF. In view of the relationship between the prime ideals in R and those in the total quotient ring of R, (1) implies that every prime ideal in K is maximal. By the proposition, this means that $K_M = g_M(K)$ is its own total quotient ring, and so (2) holds.

Conversely, if $P<Q$ are prime ideals in R consisting entirely of zerodivisors, and if $f \in Q-P$ then $PK<QK$ are proper prime ideals in K, $f \in QK-PK$, and, as in the proof of Proposition 1, $g_f(K) < K_f \subseteq$ total quotient ring of R_f; hence (2) does not hold.

EXAMPLES. The conditions of the corollary are satisfied by any noetherian ring in which the ideal (0) has no embedded primes; in particular, by any noetherian ring without nilpotent elements [**3**, §§5–6 of Chapter IV].

Another example is the following: Let k be a field, R and R' two integral domains containing k. Then $R \otimes_k R'$ satisfies the conditions [**3**, p. 191].

The conductor. Let B be a ring, let A be a subring of B, and let $\bar{A}$ be the integral closure of A in B. If M is a multiplicative system in A, then, after suitable identifications are made, $A_M \subseteq \bar{A}_M \subseteq B_M$, and by [**2**, p. 22] $\bar{A}_M$ is the integral closure of A_M in B_M.

The *conductor* $\mathfrak{C}(A, B)$ of A in B is defined to be the annihilator of the A-module $\bar{A}/A$. $\mathfrak{C}(A, B)$ is an ideal in A. If $c \in \mathfrak{C}(A, B)$ then $c\bar{A} \subseteq A$, whence $c\bar{A}_M \subseteq A_M$, so that $c \in \mathfrak{C}(A_M, B_M)$; thus $\mathfrak{C}(A, B) \cdot A_M \subseteq \mathfrak{C}(A_M, B_M)$. In particular if $\mathfrak{C}(A, B) \cdot A_M = A_M$ then $\mathfrak{C}(A_M, B_M) = A_M$; hence *if* $\mathfrak{C}(A, B)$ *meets* M *then* A_M *is integrally closed in* B_M.

COROLLARY 2. *If R is a ring in which every prime ideal consisting of zerodivisors is minimal, if K is the total quotient ring of R, and if M is a multiplicative system in R such that M meets the conductor $\mathfrak{C}(R, K)$, then R_M is integrally closed in its total quotient ring.*

PROOF. By Proposition 1 and Corollary 1, $K_M = g_M(K) =$ the total quotient ring of R_M. We may therefore apply the preceding remarks with $A = R$, $B = K$.

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