$U(\epsilon)$-SETS FOR WALSH SERIES

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1. Let $\epsilon=(\epsilon_0, \epsilon_1, \cdots, \epsilon_n, \cdots)$ be a monotonic decreasing sequence of positive numbers tending to zero, and let $\psi_k(x)$ designate the kth Walsh function. We say a set E contained in $[0, 1)$ is a $U(\epsilon)$-set if

$$\text{(i)} \qquad |c_k| \leqq \epsilon_k, \qquad k = 0, 1, 2, \cdots$$

and

$$\text{(ii)} \qquad \lim_{n\to\infty} \sum_{k=0}^{n} c_k\psi_k(x) = 0 \text{ in } [0, 1) - E$$

implies that $c_k=0$ for $k=0, 1, 2, \cdots$.

With $|E|$ designating the Lebesgue measure of E, we intend to prove the following theorem:

THEOREM. *Given an ϵ-sequence and a δ with $0<\delta<1$, there exists a $U(\epsilon)$-set E with $|E|>1-\delta$.*

The analogue of this theorem for trigonometric series can be found in [**3**, p. 351], and the proof we shall give here will have points in common with this last-named reference.

2. We shall first state some known results concerning Walsh series as lemmas. In order to do this let us recall some facts concerning Walsh and Rademacher functions.

The Rademacher functions are defined by

$$\text{(1)} \qquad \begin{gathered} \phi_0(x) = 1 \quad (0 \leqq x < \tfrac{1}{2}), \qquad \phi_0(x) = -1 \quad (\tfrac{1}{2} \leqq x < 1), \\ \phi_0(x+1) = \phi_0(x), \qquad \phi_n(x) = \phi_0(2^n x), \quad n = 0, 1, 2, \cdots. \end{gathered}$$

The Walsh functions are then given by

$$\text{(2)} \qquad \psi_0(x) \equiv 1, \qquad \psi_k(x) = \phi_{k_1}(x) \cdots \phi_{k_r}(x)$$

for $k=2^{k_1}+\cdots+2^{k_r}$ where the non-negative integers $k_1, \cdots, k_r$ are uniquely determined by $k_{i+1}<k_i$.

We observe from (1) and (2) that for every pair of non-negative integers k and n,

$$\text{(3)} \qquad \psi_{k2^n}(x) = \psi_k(2^n x).$$

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Now both Fine [1] and Sneider [2] have established the following result, which we shall state as Lemma 1.

LEMMA 1. *If* $\lim_{n\to\infty} \sum_{k=0}^{n} c_k\psi_k(x) = 0$ *except possibly for a countable number of points in* $[0, 1)$, *then* $c_k = 0$ *for* $k = 0, 1, 2, \cdots$.

Next, we define the non-negative integer $j * k$ by

$$\psi_j(x)\psi_k(x) = \psi_{j*k}(x). \tag{4}$$

An easy computation shows that

$$\text{if } 0 \leqq j \leqq 2^M - 1 \quad \text{and} \quad q2^M \leqq k \leqq (q+1)2^M - 1, \tag{5}$$

then

$$q2^M \leqq j * k \leqq (q+1)2^M - 1.$$

Now if $\eta(x)$ is a Walsh polynomial $\eta(x) = \sum_{j=0}^{N} a_j\psi_j(x)$ and S is a Walsh series $S = \sum_{j=0}^{\infty} c_j\psi_j(x)$ with $c_j = o(1)$ as $j \to \infty$, we define the formal product $T = \eta(x)S$ to be the series

$$T = \sum_{j=0}^{\infty} b_j\psi_j(x) \quad \text{where} \quad b_j = \sum_{k=0}^{N} a_k c_{j*k}. \tag{6}$$

Sneider [2] has established the following result concerning formal products which we state as Lemma 2.

LEMMA 2.

$$\lim_{n\to\infty} \left[\sum_{j=0}^{n} b_j\psi_j(x) - \eta(x) \sum_{j=0}^{n} c_j\psi_j(x) \right] = 0$$

uniformly for x in $[0, 1)$.

3. We now prove the theorem. We suppose we are given $\epsilon_0 \geqq \epsilon_1 \geqq \cdots \geqq \epsilon_n \geqq \cdots$ with $\epsilon_n \to 0$ and δ with $0 < \delta < 1$.

We first select a monotonic increasing sequence of positive integers, $\{m_n\}_{n=1}^{\infty}$, $m_1 \leqq m_2 \leqq \cdots$, with the property

$$2^{m_n}\epsilon_{2^n} \to 0 \quad \text{and} \quad m_n \to \infty \quad \text{as} \quad n \to \infty. \tag{7}$$

Next, proceeding in a similar manner to [3, p. 351], we let $\langle x \rangle = x - [x]$ where as usual $[x]$ denotes the largest integer less than or equal to x, and define the set E_n contained in $[0, 1)$ as follows:

$$E_n = \{x \mid x \text{ in } [0, 1), \langle 2^n x \rangle \geqq 2^{-m_n}\}. \tag{8}$$

Then E_n is a set of measure $1 - 2^{-m_n}$, and by (7), $|E_n| \to 1$ as $n \to \infty$.

Using this fact, we choose n_1 such that $|E_{n_1}| > 1 - \delta 2^{-1}$. Then, we choose an integer p such that $(1 - \delta 2^{-1}) \prod_{i=1}^{\infty} (1 - p^{-i}) > (1 - \delta)$. We set $F_1 = E_{n_1}$ and determine a monotonic strictly increasing sequence of integers, $\{n_i\}_{i=1}^{\infty}$, such that $|F_{i+1}| \geqq (1-p^{-i})|F_i|$ where $F_i = E_{n_1} \cdots E_{n_i}$. We then set

$$E = E_{n_1}E_{n_2} \cdots E_{n_i} \cdots, \tag{9}$$

and observe that $|E| > 1-\delta$.

We now propose to show that E is a $U(\epsilon)$-set. To do this we set

$$p_i = 2^{n_i} \quad \text{and} \quad q_i = 2^{mn_i} \qquad i = 1, 2, \cdots \tag{10}$$

and designate the characteristic function of the interval $[0, q_i^{-1})$ extended by periodicity of period one to the whole real line by $\lambda_i(x)$. Then using the fact that $\{\psi_j(x)\}_{j=1}^{\infty}$ forms a complete orthonormal system on the interval $[0, 1)$, we see from (1) and (2) that

$$\lambda_i(x) = q_i^{-1} \sum_{j=0}^{q_i-1} \psi_j(x), \qquad i = 1, 2, \cdots. \tag{11}$$

But then from (3), (10), and (11) we obtain that

$$\lambda_i(p_ix) = q_i^{-1} \sum_{j=0}^{q_i-1} \psi_{jp_i}(x), \qquad i = 1, 2, \cdots. \tag{12}$$

Now let $S = \sum_{j=1}^{\infty} c_j\psi_j(x)$ and suppose that

$$|c_j| \leqq \epsilon_j, \qquad j = 0, 1, \cdots \tag{13}$$

and

$$S_n(x) = \sum_{j=0}^{n} c_j\psi_j(x) \to 0 \text{ for } x \text{ in } [0, 1) - E \text{ as } n \to \infty. \tag{14}$$

The proof of the theorem will be complete when we show

$$c_j = 0 \quad \text{for} \quad j = 0, 1, \cdots. \tag{15}$$

To do this, we define the formal product $T^i = \lambda_i(p_ix)S$ as in (6) and obtain from (12) that $T^i = \sum_{j=0}^{\infty} b_j^i\psi_j(x)$ where

$$b_j^i = q_i^{-1} \sum_{k=0}^{q_i-1} c_{j*kp_i}. \tag{16}$$

Now by Lemma 2 for fixed i, as $n \to \infty$,

$$\sum_{j=0}^{n} b_j^i\psi_j(x) - \lambda_i(p_ix) \sum_{j=0}^{n} c_j\psi_j(x) \to 0 \quad \text{for } x \text{ in } [0, 1). \tag{17}$$

From (14) and (17), we conclude

$$\sum_{j=0}^{n} b_j^i \psi_j(x) \to 0 \quad \text{for } x \text{ in } [0, 1) - E \text{ as } n \to \infty. \tag{18}$$

On the other hand, if x is in E, then by (9) x is in E_{n_i}. But then by (8) and (10), $p_i x$ is not in $[0, q_i^{-1})$ mod 1. Consequently, $\lambda_i(p_i x) = 0$. We conclude from (17) that

$$\sum_{j=0}^{n} b_j^i \psi_j(x) \to 0 \quad \text{for } x \text{ in } E \text{ as } n \to \infty. \tag{19}$$

We obtain, therefore, from (18), (19), and Lemma 1 that

$$b_j^i = 0 \quad \text{for} \quad j = 0, 1, \cdots \quad \text{and} \quad i = 1, 2, \cdots. \tag{20}$$

To establish (15), let j be fixed. From (4) we see that $j * 0 = j$, and therefore, we obtain from (16) and (20) that

$$c_j = - \sum_{k=1}^{q_i - 1} c_{j * k p_i}, \qquad i = 1, 2, \cdots. \tag{21}$$

From (5) and (10), we see that there is an i_0 such that for $i \geqq i_0$ and $k \geqq 1$, $j * k p_i \geqq p_i$. Since $\{\epsilon_n\}_{n=1}^{\infty}$ is a monotonic decreasing sequence, we conclude from (13) and (21) that for $i \geqq i_0$, $|c_j| \leqq q_i \epsilon_{p_i}$. But by (10) and (7), $q_i \epsilon_{p_i} \to 0$ as $i \to \infty$. Consequently, $c_j = 0$, (15) is established, and the proof to the theorem is complete.

Bibliography

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