A CHOQUET BOUNDARY FOR THE PRODUCT OF TWO COMPACT SPACES

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Let X be a compact Hausdorff space and $C(X)$ the Banach space of all real-valued continuous functions on X under the sup norm. If H is a linear subspace of $C(X)$ which separates the points of X and contains the constant functions, then it is well known that there exists a smallest closed subset of X, called the Šilov boundary of X relative to H and denoted by $\partial_H X$, with the property that each $h \in H$ attains its maximum value on $\partial_H X$. A point $x \in X$ is called an H-extremal point if the only positive linear functional u on $C(X)$ such that $u(h) = h(x)$ for all $h \in H$ is the evaluation functional ϕ_x where $\phi_x(f) = f(x)$ for all $f \in C(X)$. The set of H-extremal points of X is called the Choquet boundary of X relative to H and will be denoted by $\nabla_H X$. H. Bauer has shown [**1**, 2.1] that the Choquet boundary of X relative to H is nonempty and the Šilov boundary of X relative to H is equal to the closure of the Choquet boundary.

Bauer has also introduced (see [**1**]) an abstract Dirichlet problem for the above setting. If $S \supset \partial_H X$ is closed and $x \in X$, then M_x^S denotes the set of positive linear functionals u on $C(S)$ such that $u(h|S) = h(x)$ for all $h \in H$. M_x^S is always nonempty. The measures in $M_x^{\partial_H X}$ are called the H-harmonic measures belonging to x. A function f in $C(X)$ is said to be H-harmonic if for every $x \in X$ and every $u \in M_x^X$ we have $u(f) = f(x)$. The set of H-harmonic functions is denoted by $\hat{H}$. The Dirichlet problem is said to be solvable for X relative to H if $\hat{H} | \partial_H X = C(\partial_H X)$. The Dirichlet problem is solvable if and only if to each x in X belongs exactly one H-harmonic measure (see [**1**, Satz 9]).

Let X_1, X_2 be compact Hausdorff spaces and H_1, H_2 separating linear subspaces of $C(X_1)$, $C(X_2)$ respectively, which contain the constant functions. Equipped with the usual product topology, the cartesian product $X_1 \times X_2$ is a compact Hausdorff space. In this paper we show that a subfamily $H_1 + H_2$ of $C(X_1 \times X_2)$ can be defined, in a natural way, for which $\nabla_{H_1+H_2} X_1 \times X_2$ exists and $\nabla_{H_1+H_2} X_1 \times X_2 = \nabla_{H_1} X_1 \times \nabla_{H_2} X_2$. We then give two theorems on the relation of the

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solvability of the Dirichlet problem for X_1 and X_2 to the solvability of the Dirichlet problem for $X_1 \times X_2$.

If H is a separating linear subspace of $C(X)$ which contains the constant functions, then H^* will denote the conjugate space of H where H is equipped with the sup norm and U_H will denote the positive face of the unit sphere of H^*, i.e., the set of positive linear functionals u on H such that $u(1)=1$. We will make use of Bishop and de Leeuw's characterization of the Choquet boundary in terms of geometric extreme points. Namely, $x \in \nabla_H X$ if and only if ϕ_x^H is an extreme point of U_H where $\phi_x^H(h)=h(x)$ for all $h \in H$ (see [**1**, Hilfssatz 8] or [**2**, Lemma 4.3]).

Definition. If $h_1 \in H_1$ and $h_2 \in H_2$, then $[h_1+h_2]$ will denote the function defined on $X_1 \times X_2$ as follows:

$$[h_1 + h_2](x, y) = h_1(x) + h_2(y) \quad \text{for all } (x, y) \in X_1 \times X_2.$$

We set $H_1+H_2 = \{[h_1+h_2]: h_1 \in H_1, h_2 \in H_2\}$.

Let $(x, y), (x_0, y_0) \in X_1 \times X_2$ and $[h_1+h_2] \in H_1+H_2$. From the inequality $|h_1(x) + h_2(y) - h_1(x_0) - h_2(y_0)| \leqq |h_1(x) - h_1(x_0)| + |h_2(y)-h_2(y_0)|$ it is evident that $H_1+H_2 \subset C(X_1 \times X_2)$. It is clear from the definition that H_1+H_2 is a linear subspace containing the constant functions. Let $(x, y) \neq (x', y')$ be elements of $X_1 \times X_2$. For definiteness we may assume $x \neq x'$. Since H_1 separates the points of X_1, there exists $h_1 \in H_1$ such that $h_1(x) \neq h_1(x')$. Then $[h_1+0](x, y) \neq [h_1+0](x', y')$. Thus, H_1+H_2 separates the points of $X_1 \times X_2$. It follows that the Šilov boundary $\partial_{H_1+H_2} X_1 \times X_2$ exists and $\mathrm{Cl}(\nabla_{H_1+H_2} X_1 \times X_2) = \partial_{H_1+H_2} X_1 \times X_2$ where $\nabla_{H_1+H_2} X_1 \times X_2$ is the Choquet boundary of $X_1 \times X_2$ relative to H_1+H_2 and Cl denotes closure.

Theorem 1. $\partial_{H_1} X_1 \times \partial_{H_2} X_2 = \partial_{H_1+H_2} X_1 \times X_2$.

Proof. Let $[h_1+h_2] \in H_1+H_2$. From the equality $\sup_{X_1 \times X_2} [h_1+h_2] = \sup_{X_1} h_1 + \sup_{X_2} h_2$ and the fact that $\partial_{H_1} X_1 \times \partial_{H_2} X_2$ is closed in $X_1 \times X_2$, it follows immediately that $\partial_{H_1+H_2} X_1 \times X_2 \subset \partial_{H_1} X_1 \times \partial_{H_2} X_2$.

Now let $(x_0, y_0) \in \partial_{H_1} X_1 \times \partial_{H_2} X_2$. Let U_{x_0} and V_{y_0} be neighborhoods of x_0 and y_0 respectively. Since $x_0 \in \partial_{H_1} X_1$ and $y_0 \in \partial_{H_2} X_2$, there exist $h_1 \in H_1$ and $h_2 \in H_2$ such that $\{x: h_1(x) = \sup_{X_1} h_1\} \subset U_{x_0}$ and $\{y: h_2(y) = \sup_{X_2} h_2\} \subset V_{y_0}$. If $(x', y') \notin U_{x_0} \times V_{y_0}$, then either $h_1(x') < \sup_{X_1} h_1$ or $h_2(y') < \sup_{X_2} h_2$. Thus

$$\begin{aligned} [h_1 + h_2](x', y') &= h_1(x') + h_2(y') < \sup_{X_1} h_1 + \sup_{X_2} h_2 \\ &= \sup_{X_1 \times X_2} [h_1 + h_2]. \end{aligned}$$

Thereby, $\{(x, y): [h_1+h_2](x, y) = \sup_{X_1 \times X_2} [h_1+h_2]\} \subset U_{x_0} \times V_{y_0}$. Now the subsets of $X_1 \times X_2$ of the form $U_{x_0} \times V_{y_0}$ where U_{x_0} and V_{y_0} are neighborhoods of x_0 and y_0 respectively, constitute a base for the neighborhood system of (x_0, y_0) in the product topology. Consequently, for every neighborhood W of (x_0, y_0) there exists $[h_1+h_2]$ in H_1+H_2 such that $\{(x, y): [h_1+h_2](x, y) = \sup_{X_1 \times X_2} [h_1+h_2]\} \subset W$. Thus, $(x_0, y_0) \in \partial_{H_1+H_2} X_1 \times X_2$ since the Šilov boundary is closed.

THEOREM 2. $\nabla_{H_1} X_1 \times \nabla_{H_2} X_2 = \nabla_{H_1+H_2} X_1 \times X_2$.

PROOF. Suppose $(x, y) \notin \nabla_{H_1+H_2} X_1 \times X_2$. Then $\phi_{(x,y)}^{H_1+H_2}$ is not an extreme point of the positive face of the unit sphere of $(H_1+H_2)^*$. Thus, there exists $u_1, u_2 \in U_{H_1+H_2}$ with $u_1 \neq u_2$ and $0 < \lambda < 1$ such that $\phi_{(x,y)}^{H_1+H_2} = \lambda u_1 + (1-\lambda) u_2$. Note that if $[h_1+h_2] \in H_1+H_2$, then $[h_1+h_2] = [h_1+0]+[0+h_2]$. Since $u_1 \neq u_2$ there exists $[h_1+h_2] \in H_1+H_2$ such that $u_1([h_1+h_2]) \neq u_2([h_1+h_2])$. But $u_1([h_1+h_2]) = u_1([h_1+0]) + u_1([0+h_2])$ and $u_2([h_1+h_2]) = u_2([h_1+0]) + u_2([0+h_2])$. Thus, either $u_1([h_1+0]) \neq u_2([h_1+0])$ or $u_1([0+h_2]) \neq u_2([0+h_2])$. We assume for definiteness $u_1([h_1+0]) \neq u_2([h_1+0])$. Define $\hat{u}_1$, $\hat{u}_2$ on H_1 as follows: $\hat{u}_1(h) = u_1([h+0])$ and $\hat{u}_2(h) = u_2([h+0])$ for all $h \in H_1$. It is evident that $\hat{u}_1$ and $\hat{u}_2$ are positive linear functionals on H_1 such that $\hat{u}_i(1) = 1$ for $i = 1, 2$. Thus, $\hat{u}_1, \hat{u}_2 \in U_{H_1}$. We have $\hat{u}_1 \neq \hat{u}_2$ since

$$\hat{u}_1(h_1) = u_1([h_1 + 0]) \neq u_2([h_1 + 0]) = \hat{u}_2(h_1).$$

Now let $h \in H_1$; then

$$\begin{aligned} \lambda \hat{u}_1(h) + (1-\lambda)\hat{u}_2(h) &= \lambda u_1([h+0]) + (1-\lambda) u_2([h+0]) \\ &= \phi_{(x,y)}^{H_1+H_2}([h+0]) = h(x) = \phi_x^{H_1}(h). \end{aligned}$$

Thus, $\phi_x^{H_1} = \lambda \hat{u}_1 + (1-\lambda)\hat{u}_2$. Thus, $\phi_x^{H_1}$ is not an extreme point of U_{H_1} and thereby $x \notin \nabla_{H_1} X_1$. The argument is similar if $u_1([0+h_2]) \neq u_2([0+h_2])$. We have shown: if $(x, y) \notin \nabla_{H_1+H_2} X_1 \times X_2$, then either $x \notin \nabla_{H_1} X_1$ or $y \notin \nabla_{H_2} X_2$. Thus, $(\nabla_{H_1+H_2} X_1 \times X_2)' \subset (\nabla_{H_1} X_1 \times \nabla_{H_2} X_2)'$ where $'$ indicates complement relative to $X_1 \times X_2$. It follows that $(\nabla_{H_1} X_1 \times \nabla_{H_2} X_2) \subset \nabla_{H_1+H_2} X_1 \times X_2$.

Suppose $(x_0, y_0) \notin \nabla_{H_1} X_1 \times \nabla_{H_2} X_2$. Then either $x_0 \notin \nabla_{H_1} X_1$ or $y_0 \notin \nabla_{H_2} X_2$. We assume for definiteness $x_0 \notin \nabla_{H_1} X_1$. Thus, there exists $u \in M_{x_0}^{X_1}(H_1)$ and $g \in C(X_1)$ such that $u(g) \neq g(x_0)$. Define ψ: $C(X_1 \times X_2) \to C(X_1)$ as follows: if $\hat{f} \in C(X_1 \times X_2)$, then $\psi(\hat{f})(x) = \hat{f}(x, y_0)$ for all $x \in X_1$. Clearly, $\psi(\hat{f}) \in C(X_1)$ and ψ is a positive preserving linear map.

Now define $\hat{u}$ on $C(X_1 \times X_2)$ by $\hat{u}(\hat{f}) = u(\psi(\hat{f}))$ for all $\hat{f} \in C(X_1 \times X_2)$. Since $\hat{u}$ is the composition of two positive linear maps, it is a positive linear functional on $C(X_1 \times X_2)$. Let $[h_1+h_2] \in H_1+H_2$. Since $h_1+h_2(y_0) \in H_1$ and $u \in M_{x_0}^{X_1}(H_1)$ we have $\hat{u}([h_1+h_2]) = u(\psi([h_1+h_2])) = u(h_1+h_2(y_0)) = h_1(x_0)+h_2(y_0) = [h_1+h_2](x_0, y_0)$. Now $u(g) \neq g(x_0)$.

Define $\hat{g}\in C(X_1\times X_2)$ by $\hat{g}(x, y)=g(x)$ for all $(x, y)\in X_1\times X_2$. $\hat{g}$ is continuous since it is the composition of g with a projection map. Then $\hat{u}(\hat{g})=u(\psi(\hat{g}))=u(g)\neq g(x_0)=\hat{g}(x_0, y_0)$. Thus, $\hat{u}(\hat{g})\neq\hat{g}(x_0, y_0)$. It follows that $(x_0, y_0)\notin\nabla_{H_1+H_2}X_1\times X_2$. We have shown that if $(x_0, y_0)\notin\nabla_{H_1}X_1\times\nabla_{H_2}X_2$, then $(x_0, y_0)\notin\nabla_{H_1+H_2}X_1\times X_2$. Thereby, $\nabla_{H_1+H_2}X_1\times X_2\subset\nabla_{H_1}X_1\times\nabla_{H_2}X_2$.

REMARK. Theorem 1 can be obtained from Theorem 2. For we have

$$\begin{aligned}\mathrm{Cl}(\nabla_{H_1}X_1)\times\mathrm{Cl}(\nabla_{H_2}X_2) &= \mathrm{Cl}(\nabla_{H_1}X_1\times\nabla_{H_2}X_2)\\ &= \mathrm{Cl}(\nabla_{H_1+H_2}X_1\times X_2).\end{aligned}$$

THEOREM 3. *If the Dirichlet problem for $X_1\times X_2$ relative to H_1+H_2 is solvable, then it is solvable for X_1 relative to H_1 and for X_2 relative to H_2.*

PROOF. We show that nonsolvability for either X_1 relative to H_1 or X_2 relative to H_2 implies nonsolvability for $X_1\times X_2$ relative to H_1+H_2. We will use the uniqueness of harmonic measures characterization of solvability of the Dirichlet problem. For definiteness we may assume that the Dirichlet problem is not solvable for X_1 relative to H_1. Then there exists $x_0\in X_1$ such that $M_{x_0}^{\partial_{H_1}X_1}(H_1)\supset\{u_1, u_2\}$ where $u_1\neq u_2$. The following argument is basically the same as the proof of $\nabla_{H_1+H_2}X_1\times X_2\subset\nabla_{H_1}X_1\times\nabla_{H_2}X_2$. The equality $\partial_{H_1}X_1\times\partial_{H_2}X_2=\partial_{H_1+H_2}X_1\times X_2$ is crucial in order that the functions introduced be well-defined. Fix $y_0\in\partial_{H_2}X_2$. Define ψ: $C(\partial_{H_1+H_2}X_1\times X_2)\to C(\partial_{H_1}X_1)$ as follows: if $\hat{f}\in C(\partial_{H_1+H_2}X_1\times X_2)$, then $\psi(\hat{f})(x)=\hat{f}(x, y_0)$ for all $x\in\partial_{H_1}X_1$. As in the proof of Theorem 7.2, ψ is a positive linear map. Now define $\hat{u}_1$, $\hat{u}_2$ on $C(\partial_{H_1+H_2}X_1\times X_2)$ by $\hat{u}_i(\hat{f})=u_i(\psi(\hat{f}))$ for all $\hat{f}\in C(\partial_{H_1+H_2}X_1\times X_2)$, $i=1, 2$. Then $\hat{u}_1$ and $\hat{u}_2$ are positive linear functionals on $C(\partial_{H_1+H_2}X_1\times X_2)$. Let $[h_1+h_2]\in H_1+H_2$. Since $h_1+h_2(y_0)\in H_1$ and $u_i\in M_{x_0}^{\partial_{H_1}X_1}(H_1)$ for $i=1, 2$, we have

$$\begin{aligned}\hat{u}_i([h_1+h_2]\mid\partial_{H_1+H_2}X_1\times X_2) &= u_i(h_1+h_2(y_0)\mid\partial_{H_1}X_1)\\ &= h_1(x_0)+h_2(y_0) = [h_1+h_2](x_0, y_0).\end{aligned}$$

Thus,

$$\hat{u}_i\in M_{(x_0,y_0)}^{\partial_{H_1+H_2}X_1\times X_2}\qquad\text{for } i=1, 2.$$

Now let $g\in C(\partial_{H_1}X_1)$ such that $u_1(g)\neq u_2(g)$. Define

$$\hat{g}\in C(\partial_{H_1+H_2}X_1\times X_2)$$

by $\hat{g}(x, y)=g(x)$ for all $(x, y)\in\partial_{H_1+H_2}X_1\times X_2$. Then $\hat{u}_1(\hat{g})=u_1(\psi(\hat{g}))=u_1(g)$ and $\hat{u}_2(\hat{g})=u_2(\psi(\hat{g}))=u_2(g)$. Therefore, $\hat{u}_1\neq\hat{u}_2$. It follows that there is more than one H_1+H_2-harmonic measure belonging to the point (x_0, y_0). Consequently, the Dirichlet problem is not solvable for $X_1 \times X_2$ relative to $H_1 + H_2$.

The next theorem is a partial converse of the already proved statement: nonsolvability of the Dirichlet problem for either X_1 relative to H_1 or X_2 relative to H_2 implies nonsolvability for $X_1 \times X_2$ relative to $H_1 + H_2$. Once again the equality $\partial_{H_1} X_1 \times \partial_{H_2} X_2 = \partial_{H_1+H_2} X_1 \times X_2$ is crucial.

THEOREM 4. *Suppose there exists* $(x_0, y_0) \in X_1 \times \partial_{H_2} X_2$ *such that*

$$M_{(x_0,y_0)}^{\partial_{H_1+H_2} X_1 \times X_2} \supset \{\hat{u}_1, \hat{u}_2\}$$

where $\hat{u}_1 \neq \hat{u}_2$ *and* $S_{\hat{u}_i} \subset \partial_{H_1} X_1 \times \{y_0\}$ *for* $i = 1, 2$, *where* $S_{\hat{u}_i}$ *denotes the support of* $\hat{u}_i$. *(In particular, the Dirichlet problem is not solvable for* $X_1 \times X_2$ *relative to* $H_1 + H_2$.*) Then the Dirichlet problem is not solvable for* X_1 *relative to* H_1.

PROOF. Define ρ: $C(\partial_{H_1} X_1) \to C(\partial_{H_1+H_2} X_1 \times X_2)$ as follows: if $f \in C(\partial_{H_1} X_1)$ then $\rho(f)(x,y) = f(x)$ for all $(x,y) \in \partial_{H_1+H_2} X_1 \times X_2$. Clearly, $\rho(f) \in C(\partial_{H_1+H_2} X_1 \times X_2)$ and ρ is a positive preserving linear map. Now define u_1, u_2 on $C(\partial_{H_1} X_1)$ by $u_i(f) = \hat{u}_i(\rho(f))$ for all $f \in C(\partial_{H_1} X_1)$ and $i = 1, 2$. Then u_1 and u_2 are positive linear functionals on $C(\partial_{H_1} X_1)$. Let $h \in H_1$. Since $[h+0] \in H_1 + H_2$ and $\hat{u}_i \in M_{(x_0,y_0)}^{\partial_{H_1+H_2} X_1 \times X_2}$ for $i = 1, 2$, we have $u_i(h \mid \partial_{H_1} X_1) = \hat{u}_i([h+0] \mid \partial_{H_1+H_2} X_1 \times X_2) = h(x_0)$. Thus, $u_i \in M_{x_0}^{\partial_{H_1} X}$ for $i = 1, 2$. Now let $\hat{g} \in C(\partial_{H_1+H_2} X_1 \times X_2)$ such that $\hat{u}_1(\hat{g}) \neq \hat{u}_2(\hat{g})$. Define $g \in C(\partial_{H_1} X_1)$ by $g(x) = \hat{g}(x, y_0)$ for all $x \in \partial_{H_1} X_1$. Clearly, $g \in C(\partial_{H_1} X_1)$. Now $u_i(g) = \hat{u}_i(\rho(g))$ for $i = 1, 2$. But $\rho(g)(x, y) = \hat{g}(x, y_0)$ for all $(x, y) \in \partial_{H_1+H_2} X_1 \times X_2$. Thus, $\rho(g) \mid S_{\hat{u}_i} = \hat{g} \mid S_{\hat{u}_i}$ for $i = 1$, 2 since $S_{\hat{u}_i} \subset \partial_{H_1} X_1 \times \{y_0\}$. Hence, $\hat{u}_i(\rho(g)) = \hat{u}_i(\hat{g})$ for $i = 1, 2$. Thus, $u_1(g) \neq u_2(g)$. Consequently, more than one H_1-harmonic measure belongs to the point x_0. Thus, the Dirichlet problem is not solvable for X_1 relative to H_1.

Of course, if there exists $(x_0, y_0) \in \partial_{H_1} X_1 \times X_2$ such that

$$M_{(x_0,y_0)}^{\partial_{H_1+H_2} X_1 \times X_2} \supset \{\hat{u}_1, \hat{u}_2\}$$

where $\hat{u}_1 \neq \hat{u}_2$ and $S_{\hat{u}_i} \subset \{x_0\} \times \partial_{H_2} X_2$, then a similar proof shows that the Dirichlet problem is not solvable for X_2 relative to H_2.

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