A COMPACT CONVEX SET IN E^3 WHOSE EXPOSED POINTS ARE OF THE FIRST CATEGORY

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1. Introduction. The extreme points of a convex set Z in a linear topological space L are those points of Z which are interior to no segment contained in Z. Among the extreme points, the exposed points are those x for which there is a continuous linear functional on L which assumes its Z-maximum exactly at x.

One of the significant properties of the set E of extreme points of Z is that its convex closure is dense in Z when Z is compact and L is locally convex. Therefore, any dense subset of E will also have this property. If L is finite dimensional, the set E_p of exposed points of Z is dense in E; there is even a similar theorem for the case where Z is compact metric [2; p. 81, Remark 7(b)].

However, when Z is compact metric, E is of the second category in itself; in fact, E is a G_δ in Z. Consequently, it has been asked if E_p has these properties. (For an account of these questions, see [3].) This is known to be true if Z is in E^2. As the title states, it is false in E^3.

As for the method of construction of this example, we shall start with a cone and pare it down to the final figure in a sequence of steps. Each step utilizes the following construction.

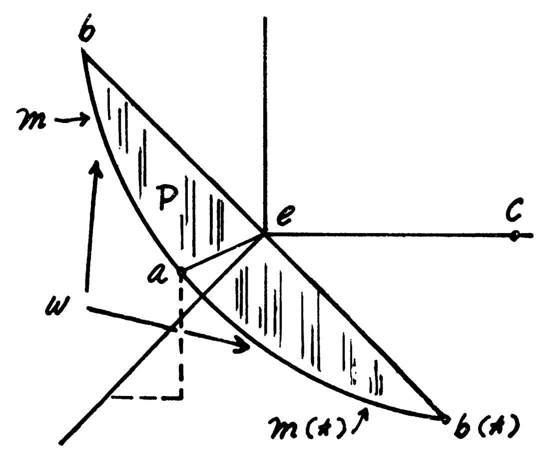

FIGURE 1

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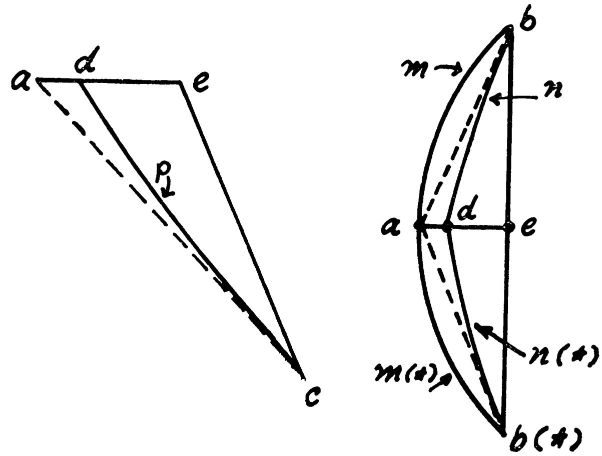
2. **A preliminary construction.** In the sequel, $[u, v]$ will denote the closed segment joining two points u and v in E^3.

At each step in §3 we will have an arc w of a circle, and a point c not in the plane P of this circle. Then the process which follows will be used at each step.

Let b and $b(*)$ denote the endpoints of w; then e will denote the midpoint of $[b(*), b]$, and a will denote the midpoint of the arc w. Let m be the subarc of w between a and b, and let $m(*)$ be the subarc between a and $b(*)$. (See Figure 1.)

In the plane determined by a, e and c, construct a circle which is tangent to $[a, c]$ at c, which contains e in its interior, and which meets $[a, e]$ at a point d not equal to a or e. In the last paragraph of this section, we will use the obvious fact that d may be chosen as close to a as we please. The shorter arc which joins c to d on this circle will be called p. (See Figure 2.)

Draw a circle in the plane P which is tangent to $[a, b]$ at b, contains e in its interior, and meets $[a, e]$ at d. Let n be the shorter arc connecting b to d. (See Figure 3.)



FIGURES 2 AND 3

For a point u on n, let $t(u)$ be the intersection with $[a, d]$ of the line tangent to n at u. Note that each point on $[a, d]$ is a $t(u)$ for a unique u on n. Also, note that there is a unique point $p(u)$ on p such that $[t(u), p(u)]$ is tangent to p at $p(u)$. (See Figure 4.)

In the interior of n, we now pick a disjoint sequence $w(1), w(2), \cdots$ of nondegenerate closed arcs such that the union of the collection of $w(i)$ is dense in n and each $w(i)$ is less than half the length of n. If

$b(i)$ and $b(i, *)$ are the endpoints of $w(i)$, let $c(i)$ be the point of intersection of the lines containing $[t(b(i)), p(b(i))]$ and $[t(b(i, *)). p(b(i, *))]$. (See Figures 4 and 5.)

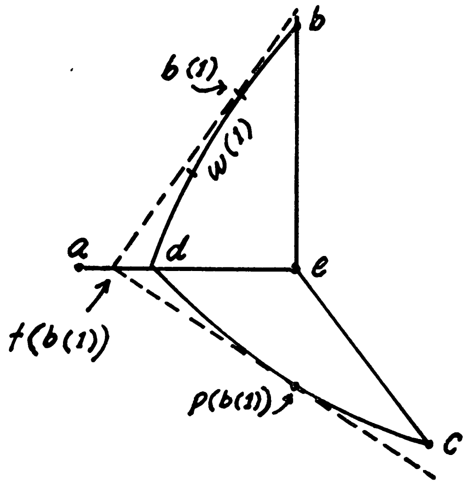
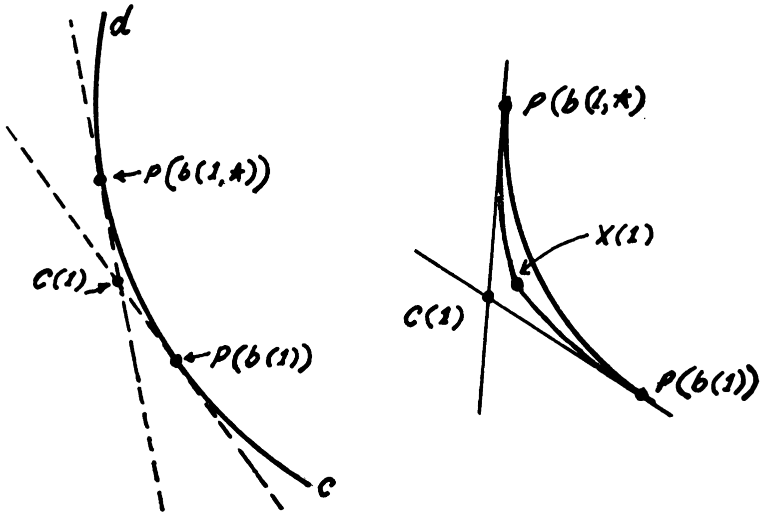


FIGURE 4



FIGURES 5 AND 6

Let q be the curve from d to c which contains all points $p(u)$ for n not in any $w(i)$ and contains all segments $[p(b(i)), c(i)]$ and $[p(b(i, *)), c(i)]$. Note that, for each u in n which is not an endpoint of any of the $w(i)$, there is a unique line through $t(u)$ which meets q at a point $q(u)$ in such a way that the line through $t(u)$ and $q(u)$ does not cross q. For $u=b(i)$ or $b(i, *)$, define $q(u)=c(i)$; then $q(u)=p(u)$ if u is not in any $w(i)$, and $q(u)=c(i)$ if u is in $w(i)$.

{An alternate construction will be made by introducing only a slight change at this point. Inside the region determined by the arc p, by $[p(b(1)), c(1)]$, and by $[p(b(1, *)), c(1)]$ construct two arcs as in Figure 6 which are on circles with half the curvature of p and which are tangent to p at $p(b(1))$ and $p(b(1, *))$, respectively. Let $x(1)$ be the point where the arcs meet. Define q_1 to be p with the arc between $p(b(1))$ and $p(b(1, *))$ replaced by the two arcs just described. Let $w'(1)$ be the points u in $w(1)$ such that $q_1(u)=x(1)$, where $q_1(u)$ is defined similarly to $q(u)$.

Let $s_1, s_2, \cdots$ be a countable dense set on n. Let us choose a possibly different $w(2)$, disjoint from $w'(1)$ such that $w(2)$ contains in its interior the first of the s_i, say s_{i_0}, which is not in $w'(1)$. We now define q_2 by repeating the above construction except that we pick the arcs with such small curvature that s_{i_0} is contained in $w'(2)$, where $w'(2)$ is defined in a manner similar to $w'(1)$.

Continuing inductively, we let the limit of the q_i's be q, and we have the union of the collection of $w'(i)$ dense in n.}

For each u in n let $H^+(u)$ be the closed half space which contains e and which is determined by the plane $H(u)$ containing u, $t(u)$ and $q(u)$.

Similarly, construct $n(*)$ with endpoint at d. Then define $t(u)$ for $n(*)$ as above, and let $w(*, i)$ be the set of u such that $t(u)=t(v)$ for some v in $w(i)$. (Note that we will need the figure in P enclosed by n, $n(*)$ and $[b(*), b]$ to be convex. This can always be accomplished by picking d close enough to a, which is possible as we have already observed.)

3. **The convex set** Z. We will now use the preliminary construction at each stage of an inductive process which will give us the desired example. Let w be half of the unit circle in the x_1x_3-plane of E^3, say $x_1 \geqq 0$ for all x in w. Let $c=(0, 1, 0)$. We perform the construction of §2 using w and c as indicated, which defines $w(i)$, $w(*, i)$ and $c(i)$.

For the remaining steps we will use a standard convention in our notation which we now describe.

At each stage we will have $w(\cdot, i)$, $w(\cdot, *, i)$ and $c(\cdot, i)$, where the dot stands for a finite (or null) sequence of symbols each of which is either a star or a positive integer. We will follow the preliminary construction for each of the pairs $w(\cdot, i)$ and $c(\cdot, i)$, and $w(\cdot, *, i)$ and

$c(\cdot, i)$. We will name the arcs and points according to their function. That is, if an object performing a certain function was called $y(j)$ or $y(*, j)$ in the preliminaries, it will now be designated $y(\cdot, i, j)$ or $y(\cdot, i, *, j)$ if we are working with $w(\cdot, i)$, and it will be $y(\cdot, *, i, j)$ or $y(\cdot, *, i, *, j)$ if we are working with $w(\cdot, *, i)$. Note that it is possible to tell the stage of the induction by counting the number of integers in parentheses after a symbol.

For example, we have defined $w(i)$, $w(*, i)$, and $c(i)$. For $w(i)$, let $b(i)$ and $b(i, *)$ be its endpoints, let $e(i)$ be the midpoint of $[b(i, *), b(i)]$, and so on. (Also, see Figure 7.)

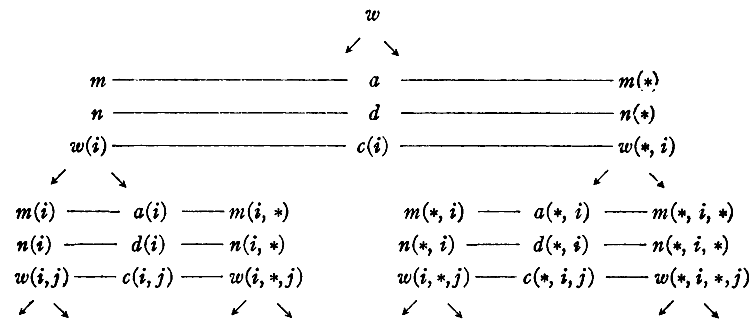


FIGURE 7

Having completed the induction, recall the definition in §2 of $H^+(u)$ where u is in n. For u in $n(\cdot)$ or $n(\cdot, *)$, we define $H^+(u)$ in the same way. That is, in the definition of $H^+(u)$ we replace n by $n(\cdot)$ or $n(\cdot, *)$, q by $q(\cdot)$, a by $a(\cdot)$ and e by $e(\cdot)$. Let A be the cone generated by w and c, and let Z be the intersection of A with each of the $H^+(u)$ for u in some $n(\cdot)$.

4. **Properties of** Z. First, it is clear that Z is compact and convex, since it is the intersection with a compact convex set of a collection of closed convex sets.

PROPERTY 1. *Each point of* $q(\cdot)\cap p(\cdot)$ *is an exposed point except points of the form* $p(b(\cdot, i))$ *and* $p(b(\cdot, i, *))$. *These latter are not exposed, except in the alternate construction where each point of* $q(\cdot)$ *is exposed. In any case, each* $c(\cdot)$ *is exposed.*

PROOF. It is useful to think of a sequence of convex sets $A = A_0, A_1, A_2, \cdots$, where A_i is the intersection with A_{i-1} of the $H^+(u)$ which are constructed at the ith step. If $H^+(u)$ is constructed at the ith step, the plane $H(u)$ which determines $H^+(u)$ meets A_{i-1} only in the cone generated by the w and c which define $H(u)$. Moreover, if u is not a $b(\cdot)$, then $H(u)$ meets A_i in a segment $L(u)$. Since each $n(\cdot)$ has a corresponding $n(\cdot, *)$, there is a u' in $n(\cdot, *)$ such that $L(u')$ meets $L(u)$ at a single point r. Every point in each $q(\cdot)$ is such a point, except as noted. Moreover, each $H^+(v)$ contains $L(u)$ and $L(u')$. Therefore $H(u)$ meets Z along $L(u)$, $H(u')$ meets Z along $L(u')$, and so r is the only point common to $H(u)$, $H(u')$, and Z.

Let f be a linear functional such that $H(u)=\{x: f(x)=1\}$ and f is not greater than 1 on $H^+(u)$. Let g be a linear functional such that $H(u')=\{x: g(x)=1\}$ and g is not greater than 1 on $H^+(u')$. The above argument shows that f assumes its Z-maximum at $L(u)$ and g assumes its Z-maximum at $L(u')$. Therefore, $f/2+g/2$ assumes its maximum exactly at r.

PROPERTY 2. *There are no exposed points other than those noted in Property* 1 *and those in the x_1x_3-plane.*

PROOF. Let r be a point which is not in the x_1x_3-plane or in the x_2x_3-plane. Each such point which is also on the boundary of Z is the limit of a sequence of points each of which is on a segment of the form $L(u)$, where $L(u)$ was defined in the proof of Property 1. Therefore r is on a segment contained in the boundary of Z and reaching to the x_1x_3 plane. If r is an extreme point, then r must be an endpoint of this segment. Let s be the other end.

If r is not on any $L(u)$, then at each step r is contained in the cone defined by some $w(\cdot, \cdot)$ and the corresponding $c(\cdot)$. Let C_i be the cone which contains r at the ith step. Then $[r, s]$ is the intersection of the collection of C_i.

Let α be the curve which starts at c, goes down q to the apex of C_1, goes down C_1 in the same way to the apex of C_2, and so on. Then α ends at r, and the part of α close to r is very nearly parallel to $[r, s]$. Therefore, r is not exposed.

The following demonstrates the theorem of the title.

PROPERTY 3. *In the alternate construction, E_p is the union of a countable number of closed sets each of which has no interior with respect to E_p. In the standard construction, E_p contains no dense G_δ.*

PROOF. Each $q(\cdot)$ is compact, and therefore closed in E_p. Moreover, $q(\cdot)$ obviously has no interior with respect to E_p since the union of the set of $q(\cdot, i)$, $i=1, 2, \cdots$, has $q(\cdot)$ in its closure. This proves the first statement; the second statement follows similarly.

The following answers a question arising from [1] where it is proved that E_p of any convex set in E^n is the union of n sets each of which is the intersection of an F_σ with a G_δ. We show that one cannot validly replace "intersection" by "union" in the above sentence.

PROPERTY 4. *In the standard construction, E_p is not the union of an F_σ with a G_δ.*

PROOF. It is easily seen that E_p is the union of a countable number of sets F_i, each of which is homeomorphic to a Cantor set with the endpoints removed and with one point added from each of the excluded open intervals used to define the Cantor set. The points corresponding to the latter will be called the "added points" of F_i. Let G_i denote F_i after the added points are removed. Since G_i is closed in E_p, any F_σ contained in E_p will intersect G_i in an F_σ. Since there are only a countable number of added points, any G_δ contained in E_p will meet the union T of the G_i in a G_δ. Therefore, if E_p is the union of an F_σ with a G_δ, then so is T.

Suppose that T is the union of F and G where the former is an F_σ and the latter is a G_δ. If F has a nonempty interior with respect to T, then $F\cap G_i$ has a nonempty interior with respect to G_i, for some i. This is impossible. Therefore, G is a dense G_δ in T. However, T is dense in E_p, and we have contradicted the last statement in Property 3. This completes the proof.

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