A CHARACTERIZATION OF THE ALMOST PERIODIC HOMEOMORPHISMS ON THE CLOSED 2-CELL

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1. Introduction. The objective of this paper is to prove that any almost periodic homeomorphism of a closed 2-cell onto itself is topologically equivalent to a reflection of a disk in a diameter or to a rotation of a disk about its center. This extends the well-known results of Kerékjártó [5] for periodic homeomorphisms (cf. Eilenberg [1, Theorem 2]).

2. A topological classification of the almost periodic homeomorphisms on a closed 2-cell. A homeomorphism $h$ of a metric space $(X, \rho)$ onto itself is said to be almost periodic on $X$ if $\epsilon > 0$ implies that there exists a relatively dense sequence \( \{n_i\} \) of integers such that $\rho(x, h^{n_i}(x)) < \epsilon$ for all $x \in X$ and $i = \pm 1, \pm 2, \cdots$. A homeomorphism $h$ of the space $X$ onto itself is said to be topologically equivalent to a homeomorphism $f$ of the space $Y$ onto itself if there exists a homeomorphism $\beta$ of $X$ onto $Y$ such that $h = \beta^{-1}f\beta$. If $h$ and $f$ are topologically equivalent, it is clear that $h$ is almost periodic on $X$ if and only if $f$ is almost periodic on $Y$. By a closed 2-cell we mean any homeomorphic image of the unit disk. With these definitions it suffices to consider almost periodic homeomorphisms on the unit disk $D$. Denote the metric in $D$ by $d(\cdot, \cdot)$.

Kerékjártó’s result [5, p. 224] for periodic homeomorphisms may be stated as follows:

**Lemma 1.** Let $f$ be a periodic homeomorphism of $D$ onto $D$. If $f$ is orientation reversing, then $f$ is topologically equivalent to a reflection of $D$ in a diameter. If $f$ is orientation preserving, then $f$ is topologically equivalent to a rotation of $D$ about its center.

Since any regularly almost periodic homeomorphism of $D$ onto $D$ is necessarily periodic [2] we have,

**Lemma 2** [4, p. 55]. Let $h$ be an almost periodic homeomorphism of $D$ onto $D$ and let $\epsilon$ be any positive number. Then there exists a periodic homeomorphism $H$ of $D$ onto $D$ such that $d(h(x), H(x)) < \epsilon$ for each $x \in D$, where $H$ may be chosen as the uniform limit of a sequence of positive powers of $h$.

A well-known characterization of the almost periodicity of $h$ is the following:

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Lemma 3 [3, p. 341]. The following are pairwise equivalent: (1) $h$ is almost periodic on $D$; (2) the set of powers of $h$ is equicontinuous; (3) the set of powers of $h$ has compact closure in the group of all homeomorphisms of $D$ onto $D$ with the usual topology; (4) there exists a compatible metric of $D$ which makes $h$ an isometry.

From (4) we see that if $h$ is almost periodic on $D$, then in the metric under which $h$ is an isometry the orbit closure of each point of $D$ lies on a metric circle about any fixed point of $D$. We will show in the nonperiodic case that each nondegenerate orbit closure is a simple closed curve and that these lie around a unique fixed point of $D$ like concentric circles.

Let $C$ denote the boundary of $D$. Then $C$ is a unit circle.

Lemma 4. If $h$ is an almost periodic homeomorphism of $D$ onto $D$ such that $h|C$ is the identity, then $h$ is the identity on $D$.

Proof. Let $\epsilon > 0$ be arbitrary. By Lemma 2 there exists a periodic homeomorphism $H$ on $D$ such that $d(h(x), H(x)) < \epsilon$ for all $x \in D$ where $H$ is the uniform limit of a sequence of positive powers of $h$. Since $h|C$ is the identity, it follows that $H|C$ is the identity. Then $H$ is periodic and orientation preserving, and hence is topologically equivalent to a rotation $r$ of $D$. Thus there exists a homeomorphism $\beta$ of $D$ onto $D$ such that $H = \beta^{-1}r\beta$. Then $r = \beta H\beta^{-1}|C$ is the identity from which it follows that $r$, and hence $H$, is the identity on $D$. Since $\epsilon > 0$ was arbitrary it follows that $h$ is the identity on $D$.

Any homeomorphism of $D$ onto $D$ is either orientation preserving or orientation reversing.

Theorem 1. If $h$ is an almost periodic orientation reversing homeomorphism of $D$ onto $D$, then $h$ is periodic of period two and hence is topologically equivalent to a reflection of $D$ in a diameter.

Proof. Using Lemma 1, it suffices to prove that $h$ is periodic of period two. Since $h|C$ is orientation reversing and almost periodic, the periodic homeomorphism $H$ of Lemma 2 is such that $H|C$ is periodic and orientation reversing. Hence $H|C$ is periodic of period two from which it follows that $h|C$ is periodic of period two. Thus $h^2|C$ is the identity and we conclude from Lemma 4 that $h^2$ is the identity on $D$. Hence $h$ is periodic of period two.

Theorem 2. Let $h$ be an almost periodic orientation preserving homeomorphism of $D$ onto $D$. Then $h$ is topologically equivalent to a rotation of $D$ through an angle $\pi \tau$, where $\tau (0 \leq \tau \leq 1)$ is uniquely determined and is rational if and only if $h$ is periodic.
Proof. If \( h \) is periodic the result is known [5] (cf. Eilenberg [1, Theorem 2]). Thus suppose \( h \) is nonperiodic. Let \( G \) be the closure of the set of integral powers of \( h \) in the group of all homeomorphisms of \( D \) onto \( D \). Then by Lemma 3, \( G \) is a compact topological group of homeomorphisms of \( D \) onto \( D \) and each \( g \in G \) is almost periodic on \( D \).

The boundary \( C \) of \( D \) is a minimal set under \( G \). Let \( x \in C \) and define \( \alpha : G \to C \) as follows: For each \( g \in G \), \( \alpha(g) = g(x) \). Then \( \alpha \) is a continuous mapping of the topological space of \( G \) onto the circle \( C \). It follows that \( \alpha \) is a homeomorphism if it is one-to-one. Thus let \( g_1, g_2 \in G \) such that \( g_1(x) = g_2(x) \). Then \( g = g_2^{-1}g_1 \in G \) is such that \( g(x) = x \).

Since \( g \) is almost periodic on \( D \), it is almost periodic on \( C \). Since \( x \) is fixed under \( g \) and \( g \) is orientation preserving it follows that \( g \vert C \) is the identity. Thus by Lemma 4, \( g \) is the identity on \( D \) and \( g_1 = g_2 \). Hence \( \alpha \) is a homeomorphism.

Thus \( G \) is a compact, connected topological group of homeomorphisms of \( D \) onto \( D \). (It follows that the character group \( G^* \) of \( G \) is an infinite cyclic group and hence that \( G \) is isomorphic to the circle group.) Since \( D \) contains a one-dimensional orbit, namely \( C \), under \( G \), all orbits in \( D \) with one exception are one-dimensional [6, p. 252]. The exceptional orbit is a fixed point \( z \) under \( G \) and there is a closed arc \( A \) from \( z \) to \( C \) such that \( A \) is a cross-section of all orbits in \( D \).

Each nondegenerate orbit is then a homogeneous, compact, and connected minimal set of dimension one. Thus each such orbit is a simple closed curve, and the family of all nondegenerate orbits lie about \( z \) like concentric circles.

The homeomorphism \( h \vert C \) is characterized by an irrational number \( \tau \) between 0 and 1, the Poincaré rotation number, and \( h \vert C \) is topologically equivalent to a rotation \( r \) of \( C \) through an angle \( \tau \pi \) [3, p. 343]. Thus there exists a homeomorphism \( \beta_0 \) of \( C \) onto \( C \) such that \( h \vert C = \beta_0^{-1} \beta \). Now let \( c \) be the endpoint of \( A \) that lies in \( C \). Define \( \beta : D \to D \) as follows: \( \beta(A) \) is a homeomorphism of \( A \) onto the radius of \( D \) to \( \beta_0(c) \) such that \( \beta(z) \) is the center of \( D \). For each \( g \in G \), \( \beta(g(A)) \) is a homeomorphism of the arc \( g(A) \) onto the radius of \( D \) to \( \beta_0(g(c)) \) such that for each \( x \in A \), \( \beta(x) \) and \( \beta(g(x)) \) lie on the same circle of \( D \) concentric with \( C \).

In order to show that \( \beta \) is well-defined we show that as \( g \) varies over \( G \) the arcs \( g(A) \) cover \( D \) and if \( g_1, g_2 \in G \) such that \( g_1 \neq g_2 \), then \( g_1(A) \) and \( g_2(A) \) have only the point \( z \) in common. It is clear that \( D \) is covered by the arcs \( g(A) \). Thus let \( x \in D - \{z\} \) such that \( g_1(x) = g_2(x) \). Then \( g = g_2^{-1}g_1 \in G \) is such that \( A \) and \( g(A) \) each go through the point \( x \). The orbit of \( x \) under \( G \) is a simple closed curve \( C' \). \( g \) is almost periodic and orientation preserving on \( C' \) and \( x \in C' \) is fixed under \( g \). Thus
$g|C'$ is the identity. The fact that $g$ is the identity on $D$ now follows by an argument similar to that used in proving Lemma 4. Thus $\beta$ is well-defined.

Since the map $\alpha(g) = g(c)$ is a homeomorphism of $G$ onto $C$, $\beta$ maps the cross-sections $g(A)$, generated by $A$, onto the radii of $D$. Since $\beta(x)$ and $\beta(g(x))$ lie on the same circle of $D$ concentric with $C$, $\beta$ maps the orbits under $G$ (the orbit closures under $h$) onto the circles of $D$ concentric with $C$. Each $x \in D - (z)$ is on but one image $g(A)$ of $A$, $g \in G$, and one of the simple closed curves formed by the orbits. Thus the map $\beta$ is one-to-one and onto. In order to show that $\beta$ is continuous and hence a homeomorphism, it suffices to show that $\beta$ is continuous at each point of $A$.

Let $x \in A$, $x \neq z$, and let $x, x' \in D$ such that $\lim_{i \to \infty} x_i = x$. Let $x_i'$ be the point of the orbit of $x_i$ under $G$ that is on $A$ and let $g_i \in G$ be such that $g_i(x_i') = x_i$. Now $\beta(x_i)$ is the intersection of the radius of $D$ to $\beta_0(g_i(c))$ and the circle of $D$ concentric with $C$ passing through $\beta(x_i')$. Since the orbit decomposition of $D$ under $G$ is continuous and $\lim_{i \to \infty} x_i = x$, $\lim_{i \to \infty} x_i' = x$, which implies that $\lim_{i \to \infty} \beta(x_i') = \beta(x)$.

Thus the concentric circles containing $\beta(x_i)$ converge to the circle containing $\beta(x)$. Now $\lim_{i \to \infty} g_i(x_i) = x$ and $\lim_{i \to \infty} x_i' = x$ imply that $\lim_{i \to \infty} g_i$ is the identity in $G$. Thus $\lim_{i \to \infty} g_i(c) = c$ implies $\lim_{i \to \infty} \beta_0(g_i(c)) = \beta_0(c)$. Hence the radii of $D$ containing $\beta(x_i)$ converge to the radius of $D$ containing $\beta(x)$. Thus $\beta$ is continuous at $x$. It is easy to see that $\beta$ is continuous at $z$. Hence $\beta$ is a homeomorphism of $D$ onto $D$. Finally $\beta|C = \beta_0$ and $h|C = \beta^{-1}R\beta$ where $r$ is a rotation of $C$ through an angle $\pi$. Then the definition of $\beta$ implies that $h = \beta^{-1}R\beta$ where $R$ is the extension of $r$ to a rotation of $D$ through an angle $\pi$. This completes the proof of the theorem.

**References**


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