A CHARACTERIZATION OF THE ALMOST PERIODIC HOMEOMORPHISMS ON THE CLOSED 2-CELL

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1. Introduction. The objective of this paper is to prove that any almost periodic homeomorphism of a closed 2-cell onto itself is topologically equivalent to a reflection of a disk in a diameter or to a rotation of a disk about its center. This extends the well-known results of Kerékjártó [**5**] for periodic homeomorphisms (cf. Eilenberg [**1**, Theorem 2]).

2. A topological classification of the almost periodic homeomorphisms on a closed 2-cell. A homeomorphism h of a metric space (X, ρ) onto itself is said to be *almost periodic* on X if $\epsilon > 0$ implies that there exists a relatively dense sequence $\{n_i\}$ of integers such that $\rho(x, h^{n_i}(x)) < \epsilon$ for all $x \in X$ and $i = \pm 1, \pm 2, \cdots$. A homeomorphism h of the space X onto itself is said to be *topologically equivalent* to a homeomorphism f of the space Y onto itself if there exists a homeomorphism β of X onto Y such that $h = \beta^{-1} f \beta$. If h and f are topologically equivalent, it is clear that h is almost periodic on X if and only if f is almost periodic on Y. By a closed 2-cell we mean any homeomorphic image of the unit disk. With these definitions it suffices to consider almost periodic homeomorphisms on the unit disk D. Denote the metric in D by $d(\cdot, \cdot)$.

Kerékjártó's result [**5**, p. 224] for periodic homeomorphisms may be stated as follows:

LEMMA 1. *Let f be a periodic homeomorphism of D onto D. If f is orientation reversing, then f is topologically equivalent to a reflection of D in a diameter. If f is orientation preserving, then f is topologically equivalent to a rotation of D about its center.*

Since any regularly almost periodic homeomorphism of D onto D is necessarily periodic [2] we have,

LEMMA 2 [**4**, p. 55]. *Let h be an almost periodic homeomorphism of D onto D and let ϵ be any positive number. Then there exists a periodic homeomorphism H of D onto D such that $d(h(x), H(x)) < \epsilon$ for each $x \in D$, where H may be chosen as the uniform limit of a sequence of positive powers of h.*

A well-known characterization of the almost periodicity of h is the following:

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LEMMA 3 [3, p. 341]. *The following are pairwise equivalent*: (1) *h is almost periodic on D*; (2) *the set of powers of h is equicontinuous*; (3) *the set of powers of h has compact closure in the group of all homeomorphisms of D onto D with the usual topology*; (4) *there exists a compatible metric of D which makes h an isometry.*

From (4) we see that if h is almost periodic on D, then in the metric under which h is an isometry the orbit closure of each point of D lies on a metric circle about any fixed point of D. We will show in the nonperiodic case that each nondegenerate orbit closure is a simple closed curve and that these lie around a unique fixed point of D like concentric circles.

Let C denote the boundary of D. Then C is a unit circle.

LEMMA 4. *If h is an almost periodic homeomorphism of D onto D such that $h|C$ is the identity, then h is the identity on D.*

PROOF. Let $\epsilon>0$ be arbitrary. By Lemma 2 there exists a periodic homeomorphism H on D such that $d(h(x), H(x))<\epsilon$ for all $x\in D$ where H is the uniform limit of a sequence of positive powers of h. Since $h|C$ is the identity, it follows that $H|C$ is the identity. Then H is periodic and orientation preserving, and hence is topologically equivalent to a rotation r of D. Thus there exists a homeomorphism β of D onto D such that $H=\beta^{-1}r\beta$. Then $r=\beta H\beta^{-1}|C$ is the identity from which it follows that r, and hence H, is the identity on D. Since $\epsilon>0$ was arbitrary it follows that h is the identity on D.

Any homeomorphism of D onto D is either orientation preserving or orientation reversing.

THEOREM 1. *If h is an almost periodic orientation reversing homeomorphism of D onto D, then h is periodic of period two and hence is topologically equivalent to a reflection of D in a diameter.*

PROOF. Using Lemma 1, it suffices to prove that h is periodic of period two. Since $h|C$ is orientation reversing and almost periodic, the periodic homeomorphism H of Lemma 2 is such that $H|C$ is periodic and orientation reversing. Hence $H|C$ is periodic of period two from which it follows that $h|C$ is periodic of period two. Thus $h^2|C$ is the identity and we conclude from Lemma 4 that h^2 is the identity on D. Hence h is periodic of period two.

THEOREM 2. *Let h be an almost periodic orientation preserving homeomorphism of D onto D. Then h is topologically equivalent to a rotation of D through an angle $\tau\pi$, where τ $(0\leqq\tau\leqq 1)$ is uniquely determined and is rational if and only if h is periodic.*

PROOF. If h is periodic the result is known [5] (cf. Eilenberg [1, Theorem 2]). Thus suppose h is nonperiodic. Let G be the closure of the set of integral powers of h in the group of all homeomorphisms of D onto D. Then by Lemma 3, G is a compact topological group of homeomorphisms of D onto D and each $g \in G$ is almost periodic on D.

The boundary C of D is a minimal set under G. Let $x \in C$ and define $\alpha: G \to C$ as follows: For each $g \in G$, $\alpha(g) = g(x)$. Then α is a continuous mapping of the topological space of G onto the circle C. It follows that α is a homeomorphism if it is one-to-one. Thus let $g_1, g_2 \in G$ such that $g_1(x) = g_2(x)$. Then $g = g_2^{-1}g_1 \in G$ is such that $g(x) = x$. Since g is almost periodic on D, it is almost periodic on C. Since x is fixed under g and g is orientation preserving it follows that $g|C$ is the identity. Thus by Lemma 4, g is the identity on D and $g_1 = g_2$. Hence α is a homeomorphism.

Thus G is a compact, connected topological group of homeomorphisms of D onto D. (It follows that the character group G^* of G is an infinite cyclic group and hence that G is isomorphic to the circle group.) Since D contains a one-dimensional orbit, namely C, under G, all orbits in D with one exception are one-dimensional [6, p. 252]. The exceptional orbit is a fixed point z under G and there is a closed arc A from z to C such that A is a cross-section of all orbits in D. Each nondegenerate orbit is then a homogeneous, compact, and connected minimal set of dimension one. Thus each such orbit is a simple closed curve, and the family of all nondegenerate orbits lie about z like concentric circles.

The homeomorphism $h|C$ is characterized by an irrational number τ between 0 and 1, the Poincaré rotation number, and $h|C$ is topologically equivalent to a rotation r of C through an angle $\tau\pi$ [3, p. 343]. Thus there exists a homeomorphism β_0 of C onto C such that $h|C = \beta_0^{-1} r \beta_0$. Now let c be the endpoint of A that lies in C. Define $\beta: D \to D$ as follows: $\beta(A)$ is a homeomorphism of A onto the radius of D to $\beta_0(c)$ such that $\beta(z)$ is the center of D. For each $g \in G$, $\beta(g(A))$ is a homeomorphism of the arc $g(A)$ onto the radius of D to $\beta_0(g(c))$ such that for each $x \in A$, $\beta(x)$ and $\beta(g(x))$ lie on the same circle of D concentric with C.

In order to show that β is well-defined we show that as g varies over G the arcs $g(A)$ cover D and if $g_1, g_2 \in G$ such that $g_1 \neq g_2$, then $g_1(A)$ and $g_2(A)$ have only the point z in common. It is clear that D is covered by the arcs $g(A)$. Thus let $x \in D - (z)$ such that $g_1(x) = g_2(x)$. Then $g = g_2^{-1}g_1 \in G$ is such that A and $g(A)$ each go through the point x. The orbit of x under G is a simple closed curve C'. g is almost periodic and orientation preserving on C' and $x \in C'$ is fixed under g. Thus

$g|C'$ is the identity. The fact that g is the identity on D now follows by an argument similar to that used in proving Lemma 4. Thus β is well-defined.

Since the map $\alpha(g) = g(c)$ is a homeomorphism of G onto C, β maps the cross-sections $g(A)$, generated by A, onto the radii of D. Since $\beta(x)$ and $\beta(g(x))$ lie on the same circle of D concentric with C, β maps the orbits under G (the orbit closures under h) onto the circles of D concentric with C. Each $x \in D-(z)$ is on but one image $g(A)$ of A, $g \in G$, and one of the simple closed curves formed by the orbits. Thus the map β is one-to-one and onto. In order to show that β is continuous and hence a homeomorphism, it suffices to show that β is continuous at each point of A.

Let $x \in A$, $x \neq z$, and let $x_i \in D$ such that $\lim_{i\to\infty} x_i = x$. Let x_i' be the point of the orbit of x_i under G that is on A and let $g_i \in G$ be such that $g_i(x_i') = x_i$. Now $\beta(x_i)$ is the intersection of the radius of D to $\beta_0(g_i(c))$ and the circle of D concentric with C passing through $\beta(x_i')$. Since the orbit decomposition of D under G is continuous and $\lim_{i\to\infty} x_i = x$, $\lim_{i\to\infty} x_i' = x$, which implies that $\lim_{i\to\infty} \beta(x_i') = \beta(x)$.

Thus the concentric circles containing $\beta(x_i)$ converge to the circle containing $\beta(x)$. Now $\lim_{i\to\infty} g_i(x_i) = x$ and $\lim_{i\to\infty} x_i' = x$ imply that $\lim_{i\to\infty} g_i$ is the identity in G. Thus $\lim_{i\to\infty} g_i(c) = c$ implies $\lim_{i\to\infty} \beta_0(g_i(c)) = \beta_0(c)$. Hence the radii of D containing $\beta(x_i)$ converge to the radius of D containing $\beta(x)$. Thus β is continuous at x. It is easy to see that β is continuous at z. Hence β is a homeomorphism of D onto D. Finally $\beta|C = \beta_0$ and $h|C = \beta_0^{-1} r \beta_0$ where r is a rotation of C through an angle $\tau\pi$. Then the definition of β implies that $h = \beta^{-1} R \beta$ where R is the extension of r to a rotation of D through an angle $\tau\pi$. This completes the proof of the theorem.

References

1. S. Eilenberg, *Sur les transformations périodiques de la surface de sphère*, Fund. Math. 22 (1934), 28–41.

2. N. E. Foland and W. R. Utz, *The embedding of discrete flows in continuous flows*, Ergodic theory, Academic Press, New York, 1963; pp. 221–234.

3. W. H. Gottschalk, *Minimal sets: an introduction to topological dynamics*, Bull. Amer. Math. Soc. 64 (1958), 336–351.

4. W. H. Gottschalk and G. A. Hedlund, *Topological dynamics*, Amer. Math. Soc. Colloq. Publ. Vol. 36, Amer. Math Soc., Providence, R. I., 1956.

5. B. V. Kerékjártó, *Vorlesungen über Topologie* I, Springer-Verlag, Berlin, 1923.

6. D. Montgomery and L. Zippin, *Topological transformation groups*, Interscience Tracts in Pure and Applied Math., No. 1, Interscience, New York, 1955.

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