## SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is normally no outlet.

# A NOTE ON A THEOREM OF I. NIVEN

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I. Niven [2] has proven the following

THEOREM. *Let $\alpha = a+2ib$ be a Gaussian integer ($b$ an integer). Then $\alpha$ can be written as a sum of two squares of Gaussian integers if and only if not both $a/2$ and $b$ are integral and odd.*

Niven's proof is based on a theorem of L. J. Mordell [1] on the representability of a binary quadratic form as a sum of two squares of linear forms. The object of this note is to give a simple direct proof of the above theorem. The proof provides explicit formulas for the representations of those Gaussian integers which can be written as a sum of two squares.

PROOF. The condition $a/2$ and $b$ not both integral and odd is necessary. For suppose $\alpha = 2a'+2ib$ with $a'$ and $b$ odd can be written in the form $\alpha = (x+iy)^2+(z+iw)^2$. This gives the equations

$$x^2 - y^2 + z^2 - w^2 = 2a', \tag{1}$$

$$xy + zw = b. \tag{2}$$

Since $b$ is odd (2) implies that either exactly two or exactly three of $x$, $y$, $z$, and $w$ are odd. (1) rules out the second possibility. Therefore exactly two of $x$, $y$, $z$, and $w$ are odd. Furthermore (2) implies that either $x$ and $y$ are both odd or $z$ and $w$ are both odd. In either case the left-hand side of (1) is divisible by 4 contradicting the fact that $a'$ is odd.

To prove sufficiency a list of formulas will be given expressing $\alpha = a+2ib$ as a sum of squares for the various cases when $a/2$ and $b$ are not both integral and odd. In each case it is easy to verify that the expressions on the right are Gaussian integers. For example, in 2. $\alpha' \pm i$ has both real and imaginary coordinates odd and therefore is divisible by $1+i$.

1. If $a$ is odd then

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$$\alpha = [(\alpha + 1)/2]^2 + i^2[(\alpha - 1)/2]^2.$$

2. If $a = 2a'$ with $a'$ odd and $b$ is even set $\alpha' = a' + ib$. Then

$$\alpha = [(\alpha' + i)/(1 + i)]^2 + i^2[(\alpha' - i)/(1 + i)]^2.$$

3. If $a = 4a'$ and $b$ is odd set $\alpha' = b - 2ia'$. Then

$$\alpha = [(\alpha' + 1)/(1 - i)]^2 + i^2[(\alpha' - 1)/(1 - i)]^2.$$

4. If $a = 4a'$ and $b = 2b'$ set $\alpha' = 2a' + 2ib'$. Then

$$\alpha = [(\alpha' + 2)/2]^2 + i^2[(\alpha' - 2)/2]^2.$$

## References

1. L. J. Mordell, *On the representation of a binary quadratic form as a sum of squares of linear forms*, Math. Z. **35** (1932), 1–15.

2. I. Niven, *Integers of quadratic fields as sums of squares*, Trans. Amer. Math. Soc. **48** (1940), 405–417.

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# A MATRIX VERSION OF RENNIE'S GENERALIZATION OF KANTOROVICH'S INEQUALITY

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Let $A$ be a positive definite hermetian matrix with eigenvalues $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_n$. $(A - \lambda_n I)(A - \lambda_1 I)A^{-1}$, where $I$ is the unit matrix, is easily seen to be negative semi-definite since the first factor is positive semi-definite, the second negative semi-definite, the third positive definite and all three commute. Hence, for any $n$-dimensional vector $x$ of unit norm

$$(Ax, x) + \lambda_1\lambda_n(A^{-1}x, x) \leqq \lambda_1 + \lambda_n.$$

Denoting the second term on the left-hand side by $u$,

$$u(Ax, x) \leqq (\lambda_1 + \lambda_n)u - u^2 \leqq (\lambda_1 + \lambda_n)^2/4$$

which is the inequality of Kantorovich. (Cf. B. C. Rennie, *An inequality which includes that of Kantorovich*, Amer. Math. Monthly **70** (1963), 982.)

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