POWERS IN EIGHTH-GROUPS

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1. Introduction. The purpose of this paper is to give an algorithm which decides whether or not an element in an eighth-group is a power. A group $G$ is an eighth-group if it is finitely presented in the form

$$G = \langle a_1, \ldots, a_n; R_1(a_1) = 1, \ldots, R_m(a_n) = 1 \rangle,$$

where (i) each defining relator is cyclically reduced and (ii) if $B_i$ and $B_j$ are cyclic transforms of $R_i$ and $R_j$, then less than one-eighth of the length of the shorter one cancels in the product $B_i^{k_1}B_j^{k_2}$, unless the product is unity. The notation in this paper is the same as that in [3]. Note that Lemma 3 and Lemma 4 in [3] hold for eighth-groups.

Reinhart [4] gives an algorithm to decide, among other things, whether or not elements in certain Fuchsian groups are powers. Note that the Fuchsian group $F(p; n_1, \ldots, n_d; m)$, see Greenberg [1], is an eighth-group if

$$4p + d + m, n_1, \ldots, n_d > 8.$$

Hence our algorithm holds for a somewhat wider class of groups and, furthermore, is purely algebraic.

**Remark.** Given any word $V$ in a finitely presented group, it is possible to find a cyclically fully reduced word $V^*$ conjugate to $V$ by writing the word $V$ in a circle and then reducing. Such a word $V^*$ will be called a reduced cyclic transform of $V$.

2. The algorithm. First we prove a lemma about eighth-groups $G$. Here $r$ denotes the length of the largest defining relator in $G$. 

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Lemma. Let $W$ be cyclically fully reduced, let $W$ be conjugate to $V$, and let $l(V) = n$. Then $l(W) \leq r^2 + rn$.

Proof of Lemma. By Greendlinger's Basic Theorem in [2, p. 643], there exist reduced cyclic transforms $W^*$ and $V^*$ of $W$ and $V$ such that $W^* = T^{-1}V^*T$, where $l(T) < r/8$ and $l(V^*) \leq l(V)$. Hence

$$l(T^{-1}V^*T) < r/8 + n + r/8 < r + n.$$ 

Consequently, by Lemma 3 in [3],

$$l(W^*) = l(T^{-1}V^*T) = r^2 + nr.$$ 

But $W$ cyclically fully reduced implies $l(W) = l(W^*)$. Hence the lemma is true.

Suppose, now, that an arbitrary word $W \neq 1$ in an eighth-group is a power, say $W = V^m$ and $l(W) = n$. Let $A$ be a reduced cyclic transform of $V$; then $W$ is conjugate to $A^m$. Lemma 4 in [3] implies that $A^m = B$, where $B$ is cyclically fully reduced and where (i) $l(B) \geq m$, and (ii) $l(B) \geq l(A) - r$. Accordingly, our lemma above implies

$$m \leq l(B) < r^2 + nr,$$

$$l(A) \leq l(B) + r < r^2 + nr + r.$$ 

The above discussion proves the following

Theorem. Let $W \neq 1$ be an arbitrary word in an eighth-group $G$ where $l(W) = n$ and $r$ is the length of the largest defining relator in $G$. Then $W$ is a power if and only if $W$ is conjugate to $A^m$ where $m$ and $A$ satisfy (1) and (2).

Since the conjugacy problem has been solved for eighth-groups by Greendlinger in [2], and since there exist only a finite number of words in any given length, the above theorem gives us our algorithm.

Bibliography