RADII OF STAR-LIKENESS AND CLOSE-TO-CONVEXITY[1]

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1. **Introduction.** Let $\mathcal{P}$ denote the class of functions $P(z)$ that are regular and have a positive real part in the unit disc E ($|z|<1$) and that are normalized so that $P(0)=1$. We shall be concerned with two classes of univalent functions that can be expressed in terms of $P(z)$. The first class, which we denote by S_α, consists of those spiral-like analytic functions $f(z)$, regular in E and normalized so that $f(0)=0$, $f'(0)=1$, with the property that

$$\Re\left\{e^{i\alpha}\frac{zf'(z)}{f(z)}\right\} \geqq 0 \qquad (|z|<1) \tag{1.1}$$

for some fixed real number α, $|\alpha|\leqq\pi/2$. It is well known [5] that the functions $f(z)$ of S_α are univalent in E. Also for the case $\alpha=0$ the members of S_0 are starlike in E. If $f(z)\in S_\alpha$ it follows easily that we can write

$$e^{i\alpha}\frac{zf'(z)}{f(z)} = (\cos\alpha)P(z) + i\sin\alpha. \tag{1.2}$$

The second class that we shall consider and denote by C_α, consists of those close-to-convex analytic functions $F(z)$, regular in E and normalized so that $F(0)=0$, $F'(0)=1$, with the property that

$$\Re\left\{e^{i\alpha}\frac{zF'(z)}{G(z)}\right\} \geqq 0 \qquad (|z|<1) \tag{1.3}$$

for some fixed real number α, $|\alpha|\leqq\pi/2$, and for some analytic function $G(z)$, regular and starlike in E, normalized so that $G(0)=0$, $G'(0)=1$. In this case $F(z)$ is said to be close-to-convex in E relative to the convex function

$$\phi(z) = e^{-i\alpha}\int_0^z \frac{G(t)}{t}\,dt \qquad (|z|<1). \tag{1.4}$$

It is well known [2] that $F(z)$ is univalent in E if $F(z)$ is close-to-convex in E.

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It should be observed that if $f(z) \in S_\alpha$ it does not follow that $f(z)$ is necessarily close-to-convex in E. For example, if $\alpha = \pi/4$ and $f_0(z)$ is defined as the function

$$f_0(z) = z \exp\{(i - 1) \operatorname{Log}(1 - iz)\},$$

where Log denotes the principal branch of the logarithm function, then it has been shown [4] that

$$\Re\left\{e^{\pi i/4} \frac{zf_0'(z)}{f_0(z)}\right\} = \frac{1}{\sqrt{2}} \frac{1 - |z|^2}{|1 - iz|^2} > 0.$$

However, $w = f_0(z)$ maps $|z| = 1$ onto a spiral curve C, covered twice, so that $f_0(z)$ is not close-to-convex in E for this would require that the tangent to C not turn back on itself through an angle exceeding π [2].

We now define for a fixed α the radius of star-likeness for the class S_α, and also for the class C_α the radius of close-to-convexity relative to the *normalized* convex function $e^{i\alpha}\phi(z)$. Let $f \in S_\alpha$ be starlike and univalent for $|z| < \rho_\alpha(f)$ and in no larger circle. Then the radius of star-likeness for the class S_α is denoted by ρ_α and defined by the equation

$$\rho_\alpha = \liminf_{f \in S_\alpha} \rho_\alpha(f). \tag{1.5}$$

Similarly, let $F(z) \in C_\alpha$ be close-to-convex relative to the normalized convex function $e^{i\alpha}\phi(z)$ for $|z| < R_\alpha(F, \phi)$ and in no larger circle. Then the radius of close-to-convexity relative to $e^{i\alpha}\phi(z)$ for the class C_α is denoted by $R_\alpha(\phi)$ and defined by the equation

$$R_\alpha(\phi) = \liminf_{F \in C_\alpha} R_\alpha(F, \phi). \tag{1.6}$$

It is the purpose of this note to show that for $|\alpha| < \pi/2$

$$\rho_\alpha = R_\alpha(\phi) = [|\sin \alpha| + \cos \alpha]^{-1} \geqq 2^{-1/2} = 0.707 \cdots. \tag{1.7}$$

It is of course well known [1] that every function $h(z)$, regular, univalent in E and normalized so that $h(0) = 0$, $h'(0) = 1$, is starlike for $|z| < \tanh \pi/4 = 0.65 \cdots$. More recently, J. Krzyż [3] has shown that $h(z)$ is close-to-convex relative to *some* convex function for $|z| < 0.80 \cdots$.

The trivial case $\alpha = 0$ gives $\rho_0 = R_0(\phi) = 1$ as is to be expected.

2. **Proof of (1.7).** In (1.2) we let $z = re^{i\theta}$, $P(re^{i\theta}) = u(r, \theta) + iv(r, \theta)$ and obtain

$$
\begin{aligned}
\Re\left\{\frac{zf'(z)}{f(z)}\right\} &= \Re\left[e^{-i\alpha}(\cos\alpha P(z) + i\sin\alpha)\right] \\
&= \Re\left[(\cos\alpha - i\sin\alpha)\{u\cos\alpha + i(\sin\alpha + v\cos\alpha)\}\right] \\
&= \cos\alpha(u\cos\alpha + v\sin\alpha) + \sin^2\alpha.
\end{aligned} \tag{2.1}
$$

For variable $P(z)=u+iv$ and fixed r and α, $|\alpha| \leqq \pi/2$, we shall find the minimum value of $u\cos\alpha+v\sin\alpha$.

By the Herglotz formula for $P(z)$ we have

$$
P(z) = \int_0^{2\pi} P_0(ze^{i\phi})\, d\alpha(\phi) \tag{2.2}
$$

where $P_0(z)=(1+z)(1-z)^{-1}$ and $\alpha(\phi)$ is nondecreasing in $[0, 2\pi]$ subject to the normalization $P(0)=1$. For proper choice of $\alpha(\phi)$, $P(z)$ reduces to $P_0(ze^{i\phi})$. Because of (2.2) we can confine our attention in (2.1) to the case $P(z)=P_0(ze^{i\phi})$ with variable ϕ. In the latter case we then have

$$
u = \frac{1-r^2}{1-2r\cos(\theta+\phi)+r^2}, \qquad v = \frac{2r\sin(\theta+\phi)}{1-2r\cos(\theta+\phi)+r^2},
$$

$$
u\cos\alpha + v\sin\alpha = \frac{(1-r^2)\cos\alpha + 2r\sin\alpha\sin\beta}{1-2r\cos\beta+r^2},
$$

where $\beta=\theta+\phi$.

For fixed $r<1$, and fixed α, $|\alpha| \leqq \pi/2$, we require the value of

$$
\begin{aligned}
m(r,\alpha) &= \min_\beta (u\cos\alpha + v\sin\alpha) \\
&= \min_\beta \left[\frac{(1-r^2)\cos\alpha + 2r\sin\alpha\sin\beta}{1-2r\cos\beta+r^2}\right].
\end{aligned}
$$

This value is provided by the following lemma.

Lemma 1. *Let r be a real number, $0 \leqq r < 1$. For all real α and β the following sharp inequality holds:*

$$
\frac{(1-r^2)\cos\alpha + 2r\sin\alpha\sin\beta}{1-2r\cos\beta+r^2} \geqq \frac{(1+r^2)\cos\alpha - 2r}{1-r^2}. \tag{2.3}
$$

Proof. By cross multiplication and simplification it is easily seen that the inequality (2.3) is equivalent to

$$
\begin{aligned}
&[2r - (1+r^2)\cos\alpha]\cos\beta - [(1-r^2)\sin\alpha]\sin\beta \\
&\qquad \leqq (1 - 2r\cos\alpha + r^2).
\end{aligned} \tag{2.4}
$$

For fixed r and α and variable β the maximum value of the left-hand side of (2.4) is the positive square root of the expression

$$[2r-(1+r^2)\cos\alpha]^2+[(1-r^2)\sin\alpha]^2=(1-2r\cos\alpha+r^2)^2.$$

Thus (2.4) and (2.3) are verified. It is easily seen that for given r and α there exists a β for which equality occurs in (2.4) and (2.3). Consequently $m(r, \alpha)$ is precisely the quantity on the right-hand side of the inequality (2.3).

From the lemma and (2.1) it now follows for $\cos\alpha \geqq 0$ and all real β, $0 \leqq r < 1$, that

$$\begin{aligned}
&\cos\alpha(u\cos\alpha+v\sin\alpha)+\sin^2\alpha\\
&\quad=\cos\alpha\left[\frac{(1-r^2)\cos\alpha+2r\sin\alpha\sin\beta}{1-2r\cos\beta+r^2}\right]+\sin^2\alpha\\
&\quad\geqq\cos\alpha\left[\frac{(1+r^2)\cos\alpha-2r}{1-r^2}\right]+\sin^2\alpha\\
&\quad=\frac{1-2r\cos\alpha+r^2\cos 2\alpha}{1-r^2}\\
&\quad=\frac{(1-r\cos\alpha)^2-r^2\sin^2\alpha}{1-r^2}.
\end{aligned}$$

From (2.1), (2.2) and the preceding discussion it now follows that whenever

$$\mathcal{R}\left\{e^{i\alpha}\frac{zf'(z)}{f(z)}\right\}\geqq 0 \qquad (|z|<1)$$

then

$$\mathcal{R}\left\{\frac{zf'(z)}{f(z)}\right\}\geqq\frac{(1-r\cos\alpha)^2-r^2\sin^2\alpha}{1-r^2}.$$

Thus, since $|\alpha| \leqq \pi/2$,

$$\mathcal{R}\left\{\frac{zf'(z)}{f(z)}\right\}\geqq 0 \quad \text{for } r\leqq[|\sin\alpha|+\cos\alpha]^{-1}.$$

The maximum value of $|\sin\alpha|+\cos\alpha$ is $2^{1/2}$ and occurs for $\alpha=\pm\pi/4$. Consequently $f(z)$ is always starlike in $|z|<2^{-1/2}=0.707\cdots$. Since the inequality (2.3) is sharp it follows that for a given α, $|\alpha|<\pi/2$, there exists a value of ϕ and a function $P(z)$ which determines a corresponding spiral-like function $f(z)$ for which

$$\mathfrak{R}\left\{\frac{zf'(z)}{f(z)}\right\} = 0 \quad \text{on } |z| = [|\sin\alpha| + \cos\alpha]^{-1}.$$

If $\alpha = \pm\pi/2$ we have the trivial case $f(z) \equiv z$. This completes the proof of (1.7) for the value of ρ_α. The proof of (1.7) for the value of $R_\alpha(\phi)$ is virtually the same as for ρ_α with only obvious modifications.

We summarize our results in the following two theorems.

THEOREM 1. *Let $f(z)$ be regular and univalent for $|z| < 1$ and normalized so that $f(0) = 0$, $f'(0) = 1$. For some α, $|\alpha| \leqq \pi/2$, let*

$$\mathfrak{R}\left\{e^{i\alpha}\frac{zf'(z)}{f(z)}\right\} \geqq 0 \quad \textit{for } |z| < 1.$$

Then

$$\mathfrak{R}\left\{\frac{zf'(z)}{f(z)}\right\} \geqq \frac{1 - 2r\cos\alpha + r^2\cos 2\alpha}{1 - r^2}$$

and $f(z)$ is starlike for $|z| \leqq \rho_\alpha = [|\sin\alpha| + \cos\alpha]^{-1}$. The estimate for ρ_α is sharp for each α, $|\alpha| < \pi/2$.

THEOREM 2. *Let $F(z)$ be regular and univalent for $|z| < 1$ and normalized so that $F(0) = 0$, $F'(0) = 1$. For some α, $|\alpha| \leqq \pi/2$, let*

$$\mathfrak{R}\left\{e^{i\alpha}\frac{zF'(z)}{G(z)}\right\} \geqq 0 \qquad (|z| < 1)$$

where $G(z)$ is regular and starlike for $|z| < 1$ and $G(0) = 0$, $G'(0) = 1$. Then $F(z)$ is close-to-convex relative to the normalized convex function

$$e^{i\alpha}\phi(z) = \int_0^z \frac{G(t)}{t}\,dt = z + \cdots$$

for $|z| \leqq R_\alpha(\phi) = [|\sin\alpha| + \cos\alpha]^{-1}$. The estimate for $R_\alpha(\phi)$ is sharp for each α, $|\alpha| < \pi/2$.

For reference purposes we add the following theorem that is derived by arguments similar to those used in this paper, in particular from an application of Lemma 1. Since the proof involves only minor and obvious changes we shall not include it here.

THEOREM 3. *Let $g(z)$ be analytic in $|z| < 1$ and let $g(0) = 1$. Let α and γ be real numbers subject to the inequalities $|\alpha| < \pi/2$, $|\gamma| < \pi/2$. If*

$$\mathfrak{R}\{e^{i\alpha}g(z)\} > 0 \quad \textit{in } |z| < 1$$

then for $|z| = r < 1$

$$\Re\{e^{i\gamma}g(z)\} \geqq \frac{\cos\gamma - 2r\cos\alpha + r^2\cos(2\alpha - \gamma)}{1 - r^2}.$$

This inequality is sharp for each α *and* γ.

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