# PARTIAL DIFFERENTIAL EQUATIONS AND DIFFERENCE EQUATIONS

LEOPOLD FLATTO

1. **Introduction.** Let $f(x_1, \cdots, x_n)$ be a real valued continuous function defined in an $n$-dimensional region $R$ and let it be a solution of the overdetermined system of partial differential equations

$$P_i(\partial/\partial x)f = 0 \qquad (1 \leqq i \leqq m) \tag{1.1}$$

where $x=(x_1, \cdots, x_n)$, $\partial/\partial x=(\partial/\partial x_1, \cdots, \partial/\partial x_n)$. The $P_i$'s are assumed to be homogeneous polynomials with real coefficients. The term solution is used to include the generalized solutions. A generalized solution is any function continuous on $R$ which is a uniform limit on compact subsets of $C^\infty$ solutions (see [**2**, p. 65]).

We wish to characterize those systems (1.1) for which all solutions satisfy a difference equation

$$\sum_{i=1}^{N} \mu_i f(x + ty_i) = 0, \quad x \in R, \quad 0 < t < \epsilon_x. \tag{1.2}$$

The $y_i$'s denote the vectors $(y_{i1}, \cdots, y_{in})$. The $\mu_i$'s are real numbers such that $\sum_{i=1}^N \mu_i=0$. The two dimensional wave equation

$$(\partial^2/\partial x_1^2 - \partial^2/\partial x_2^2)f = 0$$

is a well known example of such a system. In this case

$$\mu_1 = \mu_2 = 1, \ \mu_3 = \mu_4 = -1,$$
$$y_1 = (1, 0), \quad y_2 = (-1, 0),$$
$$y_3 = (0, 1), \quad y_4 = (0, -1).$$

The above described systems are characterized in §2. If the system consists of just one equation, then we obtain a geometric criterion on the discrete measure $\mu$ which guarantees that all solutions of (1.1) satisfy (1.2). We also obtain for this case a geometric criterion on $\mu$ insuring that (1.1) is equivalent to (1.2).

2. **The characterization of systems whose solutions satisfy the difference equation (1.2).** We require the following two lemmas which are of independent interest. We use the notation $x \cdot y = x_1y_1 + \cdots + x_ny_n$.

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LEMMA 1. *Let* $\mathfrak{A}$ *be a homogeneous ideal and let* $M_{\mathfrak{A}}$ *be its associated manifold of complex zeros. Let* $\sum_{i=1}^{N} \alpha_i e^{y_i \cdot x} = 0$ *whenever* $x \in M_{\mathfrak{A}}$. *Then* $\mathfrak{A}$ *contains a polynomial which factors into linear homogeneous terms. If the* $y_i$*'s are real, then the linear terms have real coefficients.*

It follows that if $\mathfrak{A} = (P)$, then $P$ splits into linear homogeneous factors.

PROOF. Let $x \in M_{\mathfrak{A}}$. Since $\mathfrak{A}$ is a homogeneous ideal $zx \in M_{\mathfrak{A}}$ for complex $z$. Hence $\sum_{i=1}^{N} \alpha_i \exp[y_i \cdot x]z = 0$ for all complex $z$. This is clearly impossible unless some of the $(y_i \cdot x)$'s are identical. Thus $R(x) = \prod_{1 \leq i < j \leq N} (y_i - y_j) \cdot x = 0$ whenever $x \in M_{\mathfrak{A}}$. It follows from Hilbert's Nullstellensatz that $R^k \in \mathfrak{A}$ for some positive integer $k$. $R^k$ is the desired polynomial.

The functions discussed in Lemma 2 and in the remainder of the paper are assumed to be real valued.

LEMMA 2. *Let* $f$ *be a* $C^\infty$ *solution of* $P(\partial/\partial x)f = 0$ *for all* $x$ *and let* $P(x) = \prod_{j=1}^{r} L_j^{k_j}(x)$ *where the* $L_j$*'s denote distinct linear homogeneous factors. Then* $f = f_1 + f_2 + \cdots + f_r$ *where* $f_j$ $(1 \leqq j \leqq r)$ *is a* $C^\infty$ *solution of* $L_j^{k_j}(\partial/\partial x)f_j = 0$.

PROOF. The proof is by induction on $r$. Assume that the lemma holds for $r-1$. Let $\prod_{j=1}^{r} L_j^{k_j}(\partial/\partial x)f = 0$ so that

$$L_r^{k_r}(\partial/\partial x)f = g_1 + g_2 + \cdots + g_{r-1} \tag{2.1}$$

where $g_j \in C^\infty$ and $L_j^{k_j}(\partial/\partial x)g_j = 0$ $(1 \leqq j \leqq r-1)$. Suppose that we have $r-1$ $C^\infty$ functions $f_1, \cdots, f_{r-1}$ where

$$L_r^{k_r}(\partial/\partial x)f_j = g_j, \qquad L_j^{k_j}(\partial/\partial x)f_j = 0 \qquad (1 \leqq j \leqq r-1). \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$L_r^{k_r}(\partial/\partial x)[f - (f_1 + \cdots + f_{r-1})] = 0. \tag{2.3}$$

Thus $f = f_1 + \cdots + f_r \in C^\infty$ and $L_r^{k_r}(\partial/\partial x)f_r = 0$. It remains to demonstrate the existence of the functions $f_1, \cdots, f_{r-1}$. We demonstrate the existence of $f_1$; the existence of $f_2, \cdots, f_{r-1}$ is shown in a similar fashion.

We choose a linear transformation $x = T\xi$ such that the equations

$$L_1^{k_1}(\partial/\partial x)g_1 = 0, \quad L_1^{k_1}(\partial/\partial x)f_1 = 0, \quad L_r^{k_r}(\partial/\partial x)f_1 = g_1 \tag{2.4}$$

are transformed into

$$\partial^{k_1} g_1/\partial \xi_1^{k_1} = 0, \quad \partial^{k_1} f_1/\partial \xi_1^{k_1} = 0, \quad \partial^{k_r} f_1/\partial \xi_2^{k_r} = g_1. \tag{2.5}$$

It follows by direct integration that $g_1 = A_1 + A_2\xi_1 + \cdots + A_{k-1}\xi_1^{k-1}$ where $A_1, A_2, \cdots, A_{k-1}$ are $C^\infty$ functions of $\xi_2, \cdots, \xi_n$. We define $f_1$ as

$$(2.6)\quad f_1(\xi_1, \cdots, \xi_n) = \sum_{i=1}^{k_1} \xi_1^{i-1} \int_0^{\xi_2} A_i(\tau, \xi_3, \cdots, \xi_n)(\xi_2 - \tau)^{k_r - 1} d\tau.$$

It is readily seen that $f_1(\xi_1, \cdots, \xi_n) \in C^\infty$ and satisfies $\partial^{k_1} f_1/\partial\xi_1^{k_1} = 0$, $\partial^{k_r} f_1/\partial\xi_2^{k_r} = g_1$.

We remark that the above result holds if $f(x)$ is defined in a sphere $|x| < \epsilon (|x| = \sqrt{(x_1^2 + \cdots + x_n^2)})$ instead of all space, the proof being the same as the one given above. It follows furthermore, from the above proof that each $f_j$ may be written as

$$(2.7)\quad f_j = \sum_{i=1}^{k} A_i(\xi_{j2}, \cdots, \xi_{jn})\xi_{j1}^{i-1}$$

where $x = T_j\xi_j$, $T_j$ being an orthogonal transformation and $\xi_{j1} = L_j(x)/(\sum_{k=1}^n a_{jk}^2)^{1/2}$, $L_j(x) = a_j \cdot x$. These remarks will be used in the proofs of theorems (2.1) and (2.2).

We now state our main result.

THEOREM 2.1. *Suppose that all solutions of* (1.1) *satisfy* (1.2) *for a fixed $x$ and a fixed $t > 0$. Then the ideal $\mathcal{P} = (P_1, \cdots, P_m)$ contains a polynomial which splits into homogeneous linear factors. Conversely if $\mathcal{P}$ contains such a polynomial then there exists a set of real numbers $\mu_1, \cdots, \mu_N$ with $\sum_{i=1}^N \mu_i = 0$ such that all solutions of* (1.1) *satisfy* (1.2).

PROOF. Let $e^{z \cdot x}$ be a solution of (1.1); i.e. $P_i(z) = 0$ $(1 \leqq i \leqq m)$. If all solutions of (1.1) satisfy (1.2) for a fixed $x$ and a fixed $t > 0$, then $\sum_{i=1}^N \mu_i e^{z \cdot ty_i} = 0$ whenever $P_i(z) = 0$ $(1 \leqq i \leqq m)$. By Lemma 1, the ideal $\mathcal{P}$ contains a polynomial which splits into real linear homogeneous factors.

Conversely let $\mathcal{P}$ contain a polynomial $P(x) = \prod_{j=1}^r L_j^{k_j}(x)$, the $L_j$'s being distinct linear factors. Then $P(\partial/\partial x)f = 0$ provided $f$ is a $C^\infty$ solution of (1.1). By Lemma 2, $f(y) = \sum_{j=1}^N f_j(y)$ for $|x - y| < \epsilon_x$, $f_j$ being a $C^\infty$ solution of $L_j^{k_j}(\partial/\partial y)f_j = 0$. Let $\Delta_j(t)f = f(x + ta_j) - f(x)$ where $L_j(x) = a_j \cdot x$ and let $g_j(s) = f_j(x + sa_j)$. Then $g_j^{k_j}(s) = L_j^{k_j}(\partial/\partial x) \cdot f(x + sa_j) = 0$ so that $g_j(s)$ is a polynomial of degree $< k_j$. It follows that $\Delta_j^{k_j}(t)f_j = 0$ for $t$ sufficiently small. Letting $\Delta(t) = \Delta_1^{k_1}(t) \cdots \Delta_j^{k_j}(t)$ we have

$$(2.8)\quad \Delta(t)f = 0$$

for $t$ sufficiently small. (2.8) can clearly be rewritten in the same form as (1.2). Since every solution is a uniform limit of $C^\infty$ solutions on every compact subset of $R$, we have all solutions of (1.1) satisfying $\Delta(t)f=0$.

In the case where system (1.1) consists of just one equation $P(\partial/\partial x)f=0$ we can easily obtain a geometric condition on the discrete measure $\mu$ which insures that all solutions satisfy (1.2). In view of Theorem (2.1) we assume $P=\prod_{j=1}^{r} L_j^{k_j}$, the $L_j$'s denoting the distinct linear homogeneous factors. We obtain the following result.

THEOREM 2.2. *The solutions of* $P(\partial/\partial x)f=0$ *satisfy* (1.2) *if and only if for any line* $l$ *perpendicular to the hyperplane* $L_j(x)=0$ $(1\leqq i\leqq r)$ *we have*

$$\sum{}' \mu_i L_j^s(y_i) = 0 \qquad (0 \leqq s \leqq k_j - 1) \tag{2.9}$$

*where the summation is extended over all* $y_i$ *which lie on* $l$. $P(\partial/\partial x)f=0$ *is equivalent to* (1.2) *if and only if* (2.9) *holds and* $\mu$ *has a nonvanishing moment of order* $\sum_{j=1}^{r} k_j$.[1]

PROOF. Suppose all solutions of $P(\partial/\partial x)f=0$ satisfy (1.2). For each $j$ $(1\leqq j\leqq r)$ we introduce an orthogonal transformation $y=T_j\xi$ where

$$\xi_1 = L_j(y) \Big/ \left(\sum_{k=1}^{n} a_{jk}^2\right)^{1/2}; \; L_j(y) = \sum_{k=1}^{n} a_{jk}y_k.$$

Let $y_i=T_j\xi_i$ $(1\leqq i\leqq N)$ and let $g(\xi)=f(x+T_j\xi)$. Equation (1.2) is transformed into

$$\sum_{i=1}^{N} \mu_i g(t\xi_i) = 0. \tag{2.10}$$

We choose $g(\xi_1, \cdots, \xi_n)=\xi_1^s K(\xi_p)$ where $K\in C^\infty$, $\xi_p=(\xi_2, \cdots, \xi_n)$, and $0\leqq s\leqq k_j-1$. $f$ will then satisfy $P(\partial/\partial x)f=0$ and (2.10) becomes

$$\sum_{i=1}^{N} \mu_i \xi_i^s K(t\xi_p) = 0. \tag{2.11}$$

Let $l_1, \cdots, l_q$ denote those lines perpendicular to the hyperplane $L_j(x)=0$ and containing at least one mass point $\mu_i$. Let $\eta_k$ denote the $\xi_p$ co-ordinate of $l_k$. For each $l_k$ $(1\leqq k\leqq q)$ choose a $K$ to be $=0$ at $t\eta_j$ $(1\leqq j\leqq q, j\neq k)$, and $=1$ at $t\eta_k$. (2.11) then becomes

$$\sum{}' \mu_i \xi_i^s = 0 \tag{2.12}$$

[1] The order of moment the $\int y_1^{\alpha_1} \cdots y_n^{\alpha_n} d\mu(y)$ is defined to be $\alpha_1+\cdots+\alpha_n$.

or equivalently

$$\sum{}' \mu_i L_j^s(y_i) = 0 \tag{2.13}$$

the summation being extended over all $y_i$ which lie on $l_k$.

Conversely suppose that (2.9) holds. Let $f$ be a $C^\infty$ function satisfying $P(\partial/\partial x)f=0$. By Lemma 2, $f(x+y)=f_1(y)+\cdots+f_r(y)$ for $|y|<\epsilon_x$ where $L_j^{k_j}(\partial/\partial y)f_j=0$ $(1\leqq j\leqq r)$. It follows from the representation (2.6) that $\sum_{i=1}^N \mu_i f_j(s+ty_i)=0$ for $0<t<\epsilon_x$. Hence $\sum_{i=1}^N \mu_i f(x+ty_j)=0$ for $0<t<\epsilon_x$. The same result is obtained for any solution by finding a sequence of $C^\infty$ functions $f_1, f_2, \cdots$, satisfying $P(\partial/\partial x)f_j=0$ $(1\leqq j<\infty)$ and tending to $f$ uniformly on compact subsets of $R$.

To prove the second part of theorem (2.2) we first remark that condition (2.9) implies $P\,|\,Q_k$ $(i\leqq k<\infty)$ where $Q_k=\sum_{i=1}^N \mu_i(x\cdot y_i)^k$. For let

$$x'=T_jx,\ y'=T_jy,\ y_i'=T_jy_i\ (1\leqq i\leqq N),\ Q_k(T_j^{-1}x')=\sum_{i=1}^N \mu_i(x',y_i')^k.$$

Condition (2.9) implies that $x_j'^{k_j}\,|\,Q_j'(x')$ or $L_j^{k_j}\,|\,Q_{k_j}$ $(1\leqq j\leqq r)$. Hence $P\,|\,Q_k$. If $\mu$ has a nonvanishing moment of order $M=\sum_{j=1}^r k_j$ then $Q_M\neq 0$ and $P\,|\,Q_M$ means that $Q_M=cP$ where $c$ is a constant $\neq 0$. If $f\in C$ and satisfies (1.2), then $f$ is a solution of $Q_k(\partial/\partial x)f=0$ $(1\leqq k<\infty)$. The proof of this is identical with the proof of theorem (2.1) in [1]. Thus if condition (2.9) holds and if $\mu$ has a nonvanishing moment of order $M$, then all solutions of (1.2) satisfy $Q_M(\partial/\partial x)f=0$ so that $P(\partial/\partial x)f=0$ and (1.1) are equivalent.

Suppose now that all moments of order $M$ vanish. Let $P(x)=\sum_{|s|=M} c_s x^s$ where $s=(s_1,\cdots,s_n)$, $|s|=s_1+\cdots+s_n$, $x^s=x_1^{s_1}\cdots x_n^{s_n}$, $s!=s_1!\cdots s_n!$. Then

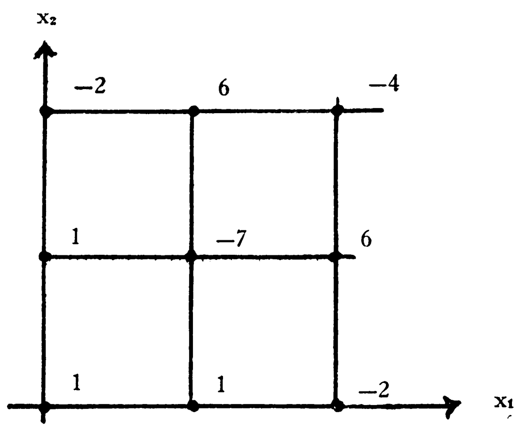
$$P(\partial/\partial x)P=\sum_{|s|=M} s!\,c_s^2\neq 0,$$

$$\sum_{i=0}^N \mu_i P(x+ty_i)=\sum_{k=0}^M t^k/k!\cdot Q_k(\partial/\partial x)f=0$$

so that (2.1) and $P(\partial/\partial x)f=0$ are not equivalent. Hence the equivalence implies condition (2.8) and the existence of a nonvanishing moment of order $M$.

Stated differently, (1.2) is equivalent to $\prod_{j=1}^r L_j^{k_j}(\partial/\partial x)f=0$ provided (2.9) holds and $\sum_{j=1}^r k_j$ is the smallest order of a nonvanishing moment of $\mu$. We illustrate this result with the following example.

Let $N=9$; let $y_i$ $(1\leqq i\leqq 9)$ denote the nine lattice points $(x_1, x_2)$ with $0\leqq x_1\leqq 2$, $0\leqq x_2\leqq 2$. The mass $\mu$ of the point $y_i$ is given by the number next to the point (see diagram). A direct calculation shows that the first nonvanishing moment is of order 3 and it can be seen by inspection that condition (2.9) is not fulfilled. Hence, in this case (1.2) is not equivalent to a single equation $P(\partial/\partial x)f=0$.

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BELFER GRADUATE SCHOOL OF SCIENCE, YESHIVA UNIVERSITY