EMBEDDING PRODUCTS OF CHAINABLE CONTINUA[1]

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Bing [1] showed that every chainable continuum can be embedded in E^2. A consequence of this is that if each of $A_1, \cdots, A_n$ is a chainable continuum, then the topological product $A_1 \times \cdots \times A_n$ can be embedded in E^{2n}. This fact can also be derived from a theorem of Isbell [2]. This paper shows that the integer $2n$ can be replaced by $n+1$. The following example shows it cannot be replaced by n.

EXAMPLE. For each integer n larger than 1 the product of $n-1$ arcs and a $\sin(1/x)$ curve cannot be embedded in E^n.

Such a continuum contains an n-cell and a subset disjoint from the n-cell with limit points in the interior of the n-cell.

McCord [4] has proved an elegant embedding theorem which will be the principal tool used here. He defines a map f from a compact subset X of a metric space (E, d) to a compact subset Y of E to be approximable by homeomorphisms (relative to E) provided that for every $\epsilon > 0$ there is an open set U containing X and a 1 to 1 map μ of U into E such that for all x in X, $d(\mu(x), f(x)) < \epsilon$.

LEMMA (THEOREM 8, CHAPTER IV OF [4].) *Let E be a compact metric space and let $\{(X_i, f_i)\}$ be an inverse sequence such that each X_i is a compact subset of E and each bonding map f_i is approximable by homeomorphisms. Then $\lim(X_i, f_i)$ can be embedded in E.*

THEOREM. *If each of $A_1, \cdots, A_n$ is a chainable continuum then $A_1 \times \cdots \times A_n$ can be embedded in E^{n+1}.*

PROOF. It is known that each nondegenerate chainable continuum is homeomorphic to the inverse limit of a sequence $\{(X_i, f_i)\}$ where each X_i is $[-1, 1]$. (See, for example, [3].) If one has inverse sequences (X_{ij}, f_{ij}), $i = 1, \cdots, n$, then $\prod_{i=1}^n \lim(X_{ij}, f_{ij})$ is homeomorphic to $\lim(\prod_{i=1}^n X_{ij}, f_{1j} \times \cdots \times f_{nj})$ where "lim" denotes inverse limit and $(f_{1j} \times \cdots \times f_{nj})(x_1, \cdots, x_n)$ is always $(f_{1j}(x_1), \cdots, f_{nj}(x_n))$. There is an inverse sequence $\{(Y_i, g_i)\}$ such that each Y_i is the n-cell $[-1, 1]^n$, each g_i is of the form $g_{1i} \times \cdots \times g_{ni}$ with each g_{ji} a map from $[-1, 1]$ into $[-1, 1]$, and such that $A_1 \times \cdots \times A_n$ is homeomorphic to $\lim(Y_i, g_i)$. The inverse sequence (Z_i, h_i), where

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each Z_i is $[-1, 1]^n \times \{0\}$ and the first n coordinates of $h_i(x_1, \cdots, x_n, 0)$ are always the first n coordinates of $g_i(x_1, \cdots, x_n)$, has an inverse limit homeomorphic to $A_1 \times \cdots \times A_n$.

Suppose a is a positive number and i is a positive integer. Define H_{ia} from $(-1-a, 1+a)^{n+1}$ into E^{n+1} by

$$\begin{aligned} &H_{ia}(x_1, \cdots, x_n, x_{n+1}) \\ &\quad = (g_{1i}(x_1/(1+a)) + (ax_2/(1+a)), g_{2i}(x_2/(1+a)) \\ &\qquad + (ax_3/(1+a)), \cdots, g_{ni}(x_n/(1+a)) + (ax_{n+1}/(1+a)), ax_1). \end{aligned}$$

A simple calculation shows that H_{ia} is a homeomorphism. For $x=(x_1, \cdots, x_n, 0)$ in Z_i the distance from $h_i(x)$ to $H_{ia}(x)$ is no more than

$$n(a/(1+a))+n \max\{ \mid g_{ji}(b)-g_{ji}(c) \mid \mid \mid b-c \mid <a/(1+a), 1 \leqq j \leqq n\}.$$

If ϵ is a positive number, there is a number a such that the distance from $H_{ia}(x)$ to $h_i(x)$ is less than ϵ for all x in Z_i. That is, each h_i is approximable by homeomorphisms. By McCord's theorem, $\lim(Z_i, h_i)$, which is homeomorphic to $A_1 \times \cdots \times A_n$, can be embedded in E^{n+1}.

The embedding theorem of McCord was proved by constructing a sequence of continua converging to a continuum homeomorphic to the continuum he wanted to embed. The author originally proved the theorem of this paper by constructing a nested sequence of $(n+1)$-cells whose intersection was homeomorphic to the product $A_1 \times \cdots \times A_n$. This less elegant method gives the additional result that the embedded continuum can be assumed cellular.

The author is indebted to Michael McCord for pointing out this improved version of the author's original theorem and a major simplification of the proof. The version of the theorem of this paper presented to the Society required the assumption that each polyhedral n-cell in E^{n+1} is collared. The author is also indebted to his major professor, W. S. Mahavier and to Lida K. Barrett for helpful discussions during the preparation of this paper.

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