EMBEDDING PRODUCTS OF CHAINABLE CONTINUA

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Bing [1] showed that every chainable continuum can be embedded in $E^2$. A consequence of this is that if each of $A_1, \ldots, A_n$ is a chainable continuum, then the topological product $A_1 \times \cdots \times A_n$ can be embedded in $E^{2n}$. This fact can also be derived from a theorem of Isbell [2]. This paper shows that the integer $2n$ can be replaced by $n+1$. The following example shows it cannot be replaced by $n$.

**Example.** For each integer $n$ larger than 1 the product of $n-1$ arcs and a $\sin(1/x)$ curve cannot be embedded in $E^n$.

Such a continuum contains an $n$-cell and a subset disjoint from the $n$-cell with limit points in the interior of the $n$-cell.

McCord [4] has proved an elegant embedding theorem which will be the principal tool used here. He defines a map $f$ from a compact subset $X$ of a metric space $(E, d)$ to a compact subset $Y$ of $E$ to be approximable by homeomorphisms (relative to $E$) provided that for every $\epsilon > 0$ there is an open set $U$ containing $X$ and a 1 to 1 map $\mu$ of $U$ into $E$ such that for all $x$ in $X$, $d(\mu(x), f(x)) < \epsilon$.

**Lemma (Theorem 8, Chapter IV of [4].)** Let $E$ be a compact metric space and let $\{(X_i, f_i)\}$ be an inverse sequence such that each $X_i$ is a compact subset of $E$ and each bonding map $f_i$ is approximable by homeomorphisms. Then $\lim(X_i, f_i)$ can be embedded in $E$.

**Theorem.** If each of $A_1, \ldots, A_n$ is a chainable continuum then $A_1 \times \cdots \times A_n$ can be embedded in $E^{n+1}$.

**Proof.** It is known that each nondegenerate chainable continuum is homeomorphic to the inverse limit of a sequence $\{(X_i, f_i)\}$ where each $X_i$ is $[-1, 1]$. (See, for example, [3].) If one has inverse sequences $(X_{ij}, f_{ij})$, $i = 1, \ldots, n$, then $\prod_{i=1}^n \lim(X_{ij}, f_{ij})$ is homeomorphic to $\lim(\prod_{i=1}^n X_{ij}, f_{ij} \times \cdots \times f_{nj})$ where "lim" denotes inverse limit and $(f_{ij} \times \cdots \times f_{nj})(x_1, \ldots, x_n)$ is always $(f_{i1}(x_1), \ldots, f_{nj}(x_n))$. There is an inverse sequence $\{(Y_i, g_i)\}$ such that each $Y_i$ is the $n$-cell $[-1, 1]^n$, each $g_i$ is of the form $g_i \times \cdots \times g_n$, with each $g_{ji}$ a map from $[-1, 1]$ into $[-1, 1]$, and such that $A_1 \times \cdots \times A_n$ is homeomorphic to $\lim(Y_i, g_i)$. The inverse sequence $(Z_i, h_i)$, where

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each $Z_i$ is $[-1, 1]^n \times \{0\}$ and the first $n$ coordinates of $h_i(x_1, \ldots, x_n, 0)$ are always the first $n$ coordinates of $g_i(x_1, \ldots, x_n)$, has an inverse limit homeomorphic to $A_1 \times \cdots \times A_n$.

Suppose $a$ is a positive number and $i$ is a positive integer. Define $H_{ia}$ from $(-1-a, 1+a)^{n+1}$ into $E^{n+1}$ by

$$H_{ia}(x_1, \ldots, x_n, x_{n+1}) = (g_{ii}(x_i/(1 + a)) + (ax_{n+1}/(1 + a)), g_{ii}(x_{n+1}/(1 + a)), \ldots, g_{ii}(x_n/(1 + a)) + (ax_n/(1 + a)), ax_i).$$

A simple calculation shows that $H_{ia}$ is a homeomorphism. For $x = (x_1, \ldots, x_n, 0)$ in $Z_i$ the distance from $h_i(x)$ to $H_{ia}(x)$ is no more than

$$n(a/(1 + a)) + n \max\{|g_{ii}(b) - g_{ii}(c)| \mid |b - c| < a/(1 + a), 1 \leq j \leq n\}.$$

If $\epsilon$ is a positive number, there is a number $a$ such that the distance from $H_{ia}(x)$ to $h_i(x)$ is less than $\epsilon$ for all $x$ in $Z_i$. That is, each $h_i$ is approximable by homeomorphisms. By McCord's theorem, $\lim(Z_i, h_i)$, which is homeomorphic to $A_1 \times \cdots \times A_n$, can be embedded in $E^{n+1}$.

The embedding theorem of McCord was proved by constructing a sequence of continua converging to a continuum homeomorphic to the continuum he wanted to embed. The author originally proved the theorem of this paper by constructing a nested sequence of $(n+1)$-cells whose intersection was homeomorphic to the product $A_1 \times \cdots \times A_n$. This less elegant method gives the additional result that the embedded continuum can be assumed cellular.

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References


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