ON POINTS OF JACOBIAN RANK k. II

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In this paper the following theorem is proved:

THEOREM 1. *Let* $f: M^n \to N^p$ *be* C^r, *where* M^n *and* N^p *are* C^r *manifolds, and let* $R_k(f)$ *be the set of points in* M^n *at which the Jacobian matrix of* f *has rank at most* k $(0 \leqq k \leqq \min(n, p))$. *Let* $i_*: \pi_m(N^p - f(R_k)) \to \pi_m(N^p)$ *be the homomorphism on the mth homotopy groups induced by the inclusion map* i. *Then* i_* *is an isomorphism (onto) for* $m+k \leqq p-2$ *and* $r \geqq \max(n-p+m+2, 1)$, *and is onto for* $m+k \leqq p-1$ *and* $r \geqq \max(n-p+m+1, 1)$.

In the previous paper [3] this theorem was proved *except* that the hypothesis "C^n", rather than "C^r, where $r \geqq \max(n-p+m+2, 1)$" etc., was used. Besides improving the differentiability hypothesis, the present proof is shorter, and shows the connection between this theorem and a theorem of Thom [**10**, p. 26]. On the other hand, this proof does not yield [**3**, (1.2)], whose proof requires almost all the lemmas of that paper.

From the examples of [**3**, p. 421, (3.3)] it follows that the differentiability hypotheses of Theorem 1 are the best possible for all n, p, and m with $0 \leqq p-n \leqq m$. A new proof of [**2**, p. 88, (1.3)] is also given (Proposition 4).

Manifolds in this note are separable, without boundary, but not necessarily connected.

REMARK 2. In his proof of [**10**, p. 26, Theorem I.5] Thom used a theorem of A. P. Morse [**10**, p. 20]; if Sard's Theorem [**9**] is used instead, the differentiability hypothesis can be changed from C^n to $C^{\max(n-q+1,1)}$. Also, in [**10**, Theorem I.6] the hypothesis C^1 suffices.

Furthermore, if r is any positive integer and H on p. 22 of [**10**] is the group of C^r diffeomorphisms rather than C^n, then Thom's proof of Theorem I.5 actually yields A a C^r diffeomorphism.

The following lemma is essentially [**3**, p. 419, (3.2)] with improved differentiability hypotheses.

LEMMA 3. *Let* $f: M^n \to N^p$ *be a* C^r *map, where* M^n *and* N^n *are* C^r *manifolds, let* k *be an integer with* $0 \leqq k \leqq \min(n, p)$, *let* Q *be a finite polyhedron with* $\dim Q \leqq p-1-k$, *and let* $r \geqq \max(n-p+\dim Q+1, 1)$. *Let* ϵ *be positive, let* $\Delta \subset Q$ *be a subpolyhedron, and let* $\alpha: Q \to N^p$ *be a*

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map such that $\alpha(\Delta)\cap f(R_k(f))=\emptyset$. *Then there exists a map* $\gamma: Q\rightarrow N^p$ *such that* $\gamma(Q)\cap f(R_k(f))=\emptyset$, $\alpha|\Delta=\gamma|\Delta$, *and the uniform distance* $d(\alpha, \gamma)<\epsilon$.

PROOF. We may suppose that M^n and N^p are C^∞ manifolds [6, p. 41]. By the Whitney Embedding Theorem N^p has a complete Riemannian metric g, and we may as well suppose that its given distance function d is that induced by g [5, p. 166, (3.5)].

Let Y be any compact set in M^n; we first prove the lemma for $R_k(f)$ replaced by $Y\cap R_k(f)$. Let $\mu=d(\alpha(\Delta), f(Y\cap R_k(f)))$, and let $V=\{x\in N^p: d(x, \alpha(\Delta))<\mu/2\}$. We may suppose that $\epsilon<\mu/2$. Since $\overline{V}$ is compact [5, p. 172], there exists ν, $0<\nu<\epsilon$, such that for each $z\in\overline{V}$ the open ball $U(z, \nu)$ with center z and radius ν is that given by [5, p. 149, (8.7)]. There exists an open neighborhood W of Δ such that $\overline{W}\neq Q$ and $\alpha(\overline{W})\subset V$.

We will define a map $\beta: Q\rightarrow N^p$ such that $d(\alpha, \beta)<\nu$ and $\beta(Q)\cap f(Y\cap R_k(f))=\emptyset$. The manifold N^p has a triangulation induced by the differential structure [6, p. 101, (10.6)]; let $\delta: Q\rightarrow N^p$ be a simplicial approximation to α with $d(\alpha, \delta)<\nu/2$. Let Γ_i $(i=1, 2, \cdots, s)$ be the (open) simplices of the polyhedron $\delta(Q)$, in order of increasing dimension.

If I is the identity diffeomorphism on N^p, there is ([10, p. 26] and Remark 2) a C^r diffeomorphism A of N^p onto itself such that the uniform distance $d(A, I)<\nu/4$ and f is transverse regular [10, p. 22] on $A^{-1}(\Gamma_1)$. Since f has rank at least $p-\dim(\Gamma_1)$ $(=p)$ at each point of $f^{-1}(A^{-1}(\Gamma_1))$ (it may be empty), and $k+1\leqq p-\dim Q$, $A^{-1}(\Gamma_1)\cap f(R_k(f))=\emptyset$. Since a C^r diffeomorphism may be approximated by a C^∞ diffeomorphism ([6, p. 39, (4.5)]; the f_1 in (4.3) and (4.5) may be chosen to approximate f), there is a C^∞ diffeomorphism B such that $d(B, I)<\nu/4$ and $B^{-1}(\Gamma_1)\cap f(Y\cap R_k(f))=\emptyset$. Let $\Psi_1=B^{-1}$.

We continue by induction. Suppose that a C^∞ diffeomorphism Ψ_i of N^p onto itself has been defined such that

$$(\mathfrak{P}_i)\ d(\Psi_i, I)<(2^{-1}-2^{-i-1})\nu \text{ and } \Psi_i\left(\bigcup_{j=1}^{i}\Gamma_j\right)\cap f(Y\cap R_k(f))=\emptyset.$$

Choose ξ such that $0<\xi<2^{-i-2}\nu$ and (since $\bigcup_{j=1}^{i}\Gamma_j$ is compact)

$$\xi<d\left(\Psi_i\left(\bigcup_{j=1}^{i}\Gamma_j\right), f(Y\cap R_k(f))\right).$$

As above, there is a C^r diffeomorphism A of N^p onto itself such that $d(A, I)<\xi$ and f is transverse regular on $A^{-1}(\Psi_i(\Gamma_{i+1}))$. Again

$$A^{-1}(\Psi_i(\Gamma_{i+1})) \cap f(Y \cap R_k(f)) = \emptyset,$$

and we may suppose that A is C^∞. Let $\Psi_{i+1}=A^{-1}\Psi_i$; Ψ_{i+1} satisfies condition $\mathfrak{P}_{i+1}$. The map $\beta=\Psi_s\delta$ satisfies the desired conditions: $d(\alpha, \beta)<\nu$ and $\beta(Q)\cap f(Y\cap R_k(f))=\emptyset$.

For each $x\in \overline{U}$, $\beta(x)\in V(\alpha(x), \nu)$. Let t be the continuous real valued function defined on Q by

$$t(x) = d(x, \Delta)[d(x, \Delta) + d(x, Q - W)]^{-1}.$$

For each $x\in W$, let $\gamma(x)$ be the (unique) point on the geodesic joining $\alpha(x)$ to $\beta(x)$ in $U(\alpha(x), \nu)$ [**5**, p. 166, Theorem 3.6] such that

$$t(x) = d(\alpha(x), \gamma(x))[d(\alpha(x), \beta(x))]^{-1};$$

for each $x\in Q-W$, let $\gamma(x)=\beta(x)$. It follows from the proof of [**5**, p. 150, Lemma 2] that $\gamma: Q\to N^p$ is continuous (if ϕ is the diffeomorphism of that lemma, then $\gamma(x)=\exp(t(x)\cdot\phi^{-1}(\alpha(x), \beta(x)))$, where $\cdot$ is scalar multiplication in E^p). Since $t=0$ on Δ, $\gamma|\Delta=\alpha|\Delta$; since $\gamma|(Q-W)=\beta|(Q-W)$, $\gamma(Q-W)\cap f(Y\cap R_k(f))=\emptyset$. Since $\gamma(x)\in U(\alpha(x), \nu)$ for each $x\in W$, and $\nu<\epsilon<\mu/2$, $d(\alpha(x), \gamma(x))<\mu/2$ for each $x\in W$; and since $\alpha(\overline{W})\subset V$, $d(\alpha(x), \alpha(\Delta))<\mu/2$ also. It follows that $\gamma(W)\cap f(Y\cap R_k(f))=\emptyset$. This completes the proof in case $R_k(f)$ is replaced by $Y\cap R_k(f)$, where Y is any compact subset of M^n.

The manifold $M^n=\bigcup_{j=1}^\infty Y_j$, where $Y_j\subset Y_{j+1}$ and Y_j is compact. Define inductively a sequence of maps γ_j: $Q\to N^p(\gamma_0=\alpha)$ such that

$$d(\gamma_j(Q), f(Y_j \cap R_k(f))) > 0$$

(call it η_j), $\gamma_j|\Delta = \gamma_{j-1}|\Delta$, and $d(\gamma_j, \gamma_{j-1}) < 2^{-j}\zeta$, where $\zeta<\min(\epsilon, \eta_i)$ ($i=1, 2, \cdots, j-1; j=1, 2, \cdots$). Since N^p is a complete metric space, the limit of the γ_j exists, call it γ; it is the desired map. To prove that $\gamma(Q) \cap f(R_k(f)) = \emptyset$, one observes that $\gamma(Q)\cap f(Y_j\cap R_k(f))=\emptyset$ ($j=1, 2, \cdots$).

Theorem 1 is an easy consequence of Lemma 3 [**3**, p. 421].

The following statement was originally proved by the author [2, p. 88, (1.3)] under the hypothesis C^n, and then by Sard [**8**, §5, Theorem 2] under the present hypothesis.

PROPOSITION 4. *If M^n and N^p are $C^{\max(n-k,1)}$ manifolds, and f: $M^n\to N^p$ is $C^{\max(n-k,1)}$, then* $\dim(f(R_k(f)))\leq k$. *In particular, if M^n and N^n are C^1 manifolds and f: $M^n\to N^n$ is C^1, then* $\dim(f(M^n))\leq n$.

We now observe that this theorem can also be obtained from Thom's theorem [**10**, p. 26].

PROOF. We may suppose that $N^p = E^p$, and prove that $\dim(f(Y\cap R_k(f)))\leq k$, where Y is any compact subset of M^n. Suppose the contrary, i.e., $\dim(f(Y\cap R_k(f)))\geqq k+1$. There exists a compact subset C of $f(Y\cap R_k(f))$ such that the homomorphism

$$i^*\colon H^k(f(Y \cap R_k(f)); Z) \to H^k(C; Z)$$

(of the Čech cohomology groups with integer coefficients) induced by inclusion is not onto [**4**, p. 151]. The Alexander Duality Theorem [**1**, p. 52] applied separately to $f(Y\cap R_k(f))$ and C yields a homomorphism

$$j_*\colon H_{p-k-1}(E^p - f(Y \cap R_k(f)); Z) \to H_{p-k-1}(E^p - C; Z)$$

(of the Čech homology groups with compact support—augmented in dimension zero) which is also not onto. Moreover j_* is induced by inclusion. (The author is grateful to F. Raymond for verifying this fact; cf. also [**7**, §5].)

Let z be a polyhedral cycle of E^p-C whose homology class γ is not in the range of j_*, and let Γ be the carrier of z. Applications of [**10**, p. 26] as in Lemma 3 yield a cycle $y\in\gamma$ with carrier $A^{-1}(\Gamma)$ disjoint from $f(Y\cap R_k(f))$. Thus γ is in the range of j_* and a contradiction results.

There is a shorter and more natural proof by induction using Thom's theorem and the inductive definition of dimension; unfortunately it requires the hypothesis C^n.

References

1. P. S. Aleksandrov, *Combinatorial topology*, Vol. 3, Graylock Press, Albany, N. Y., 1960.

2. P. T. Church, *Differentiable open maps on manifolds*, Trans. Amer. Math. Soc. **109** (1963), 87–100.

3. ———, *On points of Jacobian rank k*, Trans. Amer. Math. Soc. **110** (1964), 413–423.

4. W. Hurewicz and H. Wallman, *Dimension theory*, Princeton Univ. Press, Princeton, N. J., 1941.

5. S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Vol. I, Interscience, New York, 1963.

6. J. R. Munkres, *Elementary differential topology*, Princeton Univ. Press, Princeton, N. J., 1963.

7. F. Raymond, *Local cohomology groups with closed supports*, Math. Z. **76** (1961), 31–41.

8. A. Sard, *Hausdorff measure of critical images on Banach manifolds*, Amer. J. Math **87** (1965), 158–174.

9. ———, *The measure of the critical values of differentiable maps*, Bull. Amer. Math. Soc. **48** (1942), 883–890.

10. R. Thom, *Quelques propriétés globales des variétés différentiables*, Comment. Math. Helv. **28** (1954), 17–86.

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