COUNTEREXAMPLE TO THE MORSE-SARD THEOREM IN THE CASE OF INFINITE-DIMENSIONAL MANIFOLDS

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1. **Introduction.** The classical Morse-Sard [1], [2] theorem in the case of a function says the following: Let $f\colon \Omega \to R$ be a C^n-function, where Ω is an open set in R^n. Then, the set of critical values of f is of measure zero. Recall that $c \in R$ is a critical value of f if $f^{-1}(c)$ contains a point $x \in \Omega$ where $df_x = 0$.

As is well known, the theorem is false if we replace the assumption "$f \in C^n$" by "$f \in C^k$," where $k < n$ [3]. In case Ω is replaced by an open set in a Hilbert or Banach space B, it is easy to see that the theorem is false if either (i) f is not required to be C^∞, or (ii) B is not separable. It is only too natural to ask whether the following generalization is true: Let $F\colon \Omega \to R$ be C^∞, where Ω is an open set in a separable Hilbert space H. Then, the set of critical points for F forms a set of measure zero.

As we shall show in the following, this generalization is false. We construct a C^∞ function $f\colon H \to R$, with $H = l^2$, such that the singular set is the Cantor set $\Sigma \subset H$ formed by the sequences $(\alpha_1, \alpha_2/2, \alpha_3/3, \cdots, \alpha_n/n, \cdots)$ where, for all n, $\alpha_n = 0$ or 1. Our example also shows that the Morse-Sard theorem does not hold, in the generalized form, even if we require that the singular set be compact. The set of critical points for F will be the interval [0, 1].

2. Let H^* be the dual of H. A base of H^* is formed by the linear functions $e_1, e_2, \cdots, e_n, \cdots$, where

$$e_n(x) = x_n \quad \text{for } x = (x_1, x_2, \cdots, x_k, \cdots).$$

For any k, let $\Sigma_k(H^*)$ be the space of all k-linear symmetric continuous functions on H. $\Sigma_k(H^*)$ is also a Hilbert space, with norm $\| \ \|_k$. Clearly, $\Sigma_1(H^*) = H^*$.

To each e_n and each $k = 1, 2, 3, \cdots$, there is canonically attached an element $(e_n)^k$ in $\Sigma_k(H^*)$ defined by

$$(e_n)^k(x^1, x^2, \cdots, x^k) = e_n(x^1)e_n(x^2) \cdots e_n(x^k)$$

for any points $x^1, x^2, \cdots, x^k$ in H. Note that, for any k, $\|(e_n)^k\|_k = 1$.

To each $x \in H$, we associate a continuous linear operator $z(x)$: $\Sigma_k(H^*) \to \Sigma_{k-1}(H^*)$ for all $k \geqq 2$ by:

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For any $a \in \Sigma_k(H^*)$, let

$$z(x)(a)(x^1, x^2, \cdots, x^{k-1}) = a(x, x^1, x^2, \cdots, x^{k-1}).$$

The norm of $z(x)$ is $\|x\|$, for $\|z(x)a\|_{k-1} \leqq \|x\| \, \|a\|_k$ and $z(x)(e_n)^k = x_n(e_n)^{k-1}$.

Now, let $g: \Omega \to R$ be a function defined on the open set $\Omega \subset H$. We say that g is of class C^∞ if there exists a sequence of continuous mappings: $L^k: x \in \Omega \to L_x^k \in \Sigma_k(H^*)$ such that, for any $\epsilon > 0$ and any integer $N > 0$, there exists a $\delta > 0$ such that, if $x \in \Omega$ and $y \in H$, with $\|y\| < \delta$, then

$$| g(x+y) - g(x) - L_x^k(y) | \leqq \epsilon \|y\|$$

and

$$\| L_{x+y}^k - L_x^k - z(y) L_x^{k+1} \| < \epsilon \|y\|,$$

for $k = 1, 2, 3, \cdots, N$. The sequence $L_x^1, L_x^2, \cdots$ is then unique, and L_x^k is called the kth derivative of g at x, and is written $d^k g_x$.

3. We first introduce a C^∞-function λ: $R \to R$ with the following properties: (i) λ is an increasing C^∞-function, (ii) $-1 \leqq \lambda(t) \leqq 2$ for all t, $-\infty < t < \infty$, with $\lambda(0) = 0$, $\lambda(1) = 1$, (iii) $0 \leqq \lambda'(t) \leqq 2$, and $\lambda'(t) = 0$ only for $t = 0$ or $t = 1$, (iv) when $|t| \to \infty$, $\lambda^{(k)}(t) \to 0$ for each $k = 1, 2, \cdots$, and if $t \to \infty$, $\lambda(t) \to 2$, and if $t \to -\infty$, $\lambda(t) \to -1$. Note that there are many such functions λ. Having chosen one, let M_k be the maximum of $|\lambda^{(k)}(t)|$ for $t \in R$.

Now, define the function F: $H \to R$ as follows: For $x \in H$ $x = (x_1, x_2, \cdots)$, set

$$F(x) = \sum_{n=1}^{\infty} \frac{\lambda(n x_n)}{2^n} \cdot$$

This series is clearly absolutely convergent for all $x \in H$. We now show that the function so defined is in C^∞.

The sequence of mappings L^k: $x \in H \to L_x^k \in \Sigma_k(H^*)$ is given by

$$L_x^k = \sum_{n=1}^{\infty} \frac{n^k \lambda^{(k)}(n x_n)}{2^n} (e_n)^k.$$

This series converges and defines a member of $\Sigma_k(H^*)$ for each $x \in H$ since the series of norms $\| \; \|_k$ of its terms, i.e.,

$$\sum_{n=1}^{\infty} \frac{n^k |\lambda^{(k)}(n x_n)|}{2^n}$$

is dominated by the series $\sum_{n=1}^{\infty} n^k M_k / 2^n$, which converges. We next show that the map $x \to L_x^k$ is continuous. Take any $r \in H$. Then,

$$L_{x+r}^k - L_x^k = \sum_{n=1}^{\infty} \frac{n^k\{\lambda^{(k)}(nx_n + nr_n) - \lambda^{(k)}(nx_n)\}}{2^n}(e_n)^k.$$

By Taylor's formula,

$$\lambda^{(k)}(nx_n + nr_n) - \lambda^{(k)}(nx_n) = nr_n \int_0^1 \lambda^{(k+1)}(nx_n + tnr_n)\,dt,$$

so that

$$|\lambda^{(k)}(nx_n + nr_n) - \lambda^{(k)}(nx_n)| \leqq nr_n M_{k+1}.$$

This leads to the estimate

$$\|L_{x+r}^k - L_x^k\|_k \leqq \sum_{n=1}^{\infty} \frac{n^{k+1}M_{k+1}}{2^n} |r_n| \leqq \|r\|\sigma_{k+1},$$

where $\sigma_k = \sum_{n-1}^{\infty} n^k M_k/2^n$.

We next calculate $L_{x+r}^k - L_x^k - z(r)L_x^{k+1}$, which is easily seen to be

$$\sum_{n=1}^{\infty} \frac{n^k}{2^n}\{\lambda^{(k)}(nx_n + nr_n) - \lambda^{(k)}(nx_n) - nr_n\lambda^{(k+1)}(nx_n)\}(e_n)^k.$$

Another application of Taylor's formula to this yields the estimate $\|L_{x+r}^k - L_x^k - z(r)L_x^{k+1}\| \leqq \sigma_{k+2}\|r\|^2$. It then follows that F is of class C^∞.

Let us now determine the critical points for F. We have

$$dF_x = \sum_{n=1}^{\infty} \frac{n\lambda'(nx_n)}{2^n} e_n$$

since $(e_n)^1 = e_n$. To have dF_x identically zero, we need $\lambda'(nx_n) = 0$ for all $n = 1, 2, \cdots$. By assumption, $\lambda'(t) = 0$ only if $t = 0$ or $t = 1$. The point $x \in H$ must therefore obey: $x_n = 0$ or $1/n$, for every n. This identifies x as a member of the set Cantor-type set $\Sigma \subset H$. The critical-point set for F will be the set of points $F(x)$, for $x \in \Sigma$. Let $x = (\alpha_1, \alpha_2/2, \alpha_3/3, \alpha_4/4, \cdots)$, where each α_n is either 0 or 1. Then,

$$F(x) = \sum_1^{\infty} \frac{\lambda(nx_n)}{2^n} = \sum_{n=1}^{\infty} \frac{\lambda(\alpha_n)}{2^n}.$$

Since $\lambda(0) = 0$ and $\lambda(1) = 1$, we have $F(x) = \sum_1^\infty \alpha_n/2^n$, so that the critical-point set for F is the entire interval $[0, 1]$, which is of measure 1. We observe that the function λ could have been chosen so that not only the first, but, indeed, all the derivatives of F vanish on the set Σ.

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THE DEGREE OF APPROXIMATION BY LINEAR OPERATORS

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I. **Introduction.** In 1959 Berman [**1**] observed the following: If T denotes the unit circle, $C(T)$ the continuous functions on T, and $E_n(f)$ the error in the best approximation in the sup norm to f by a trigonometric polynomial of order n, then there could not exist a sequence of linear operators T_n mapping $C(T)$ into the trigonometric polynomials of order n, which satisfied

$$\|f - T_nf\| \leqq KE_n(f) \tag{1}$$

for some fixed constant K and $f \in C(T)$. The reason for this is easy to see. Suppose (1) is satisfied for some sequence $\{T_n\}$ and all $f \in C(T)$. If Π_n denotes the space of trigonometric polynomials of order $\leqq n$, then for $f \in \Pi_n$, $E_n(f) = 0$. Hence, $T_nf = f$ or $T_n^2 = T_n$. But by a theorem of Nikolaev [**3**, p. 494], $\|T_n\| \geqq K \log n$. Since $E_n(f) \to 0$, this is a contradiction. A similar observation holds for $L_1(T)$ where $E_n(f)$ is the best approximation by a trigonometric polynomial of order n to f in the L_1 norm.

In this note I would like to elaborate on this observation of Berman's and make some applications to Fourier series which appear to be new. Let T_n be a sequence of bounded linear operators from $C(T)$ into Π_n. In place of $E_n(f)$ let $D_n(f)$ be any continuous mapping from $C(T)$ to the non-negative reals which vanishes on Π_n. In addition to the case $D_n(f) = E_n(f)$ we may take $D_n(f) = \|f - P_nf\|$ where P_n is a bounded projection of $C(T)$ onto Π_n. Then, if $T_n^2 \neq T_n$ for each n, it

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