# ON A PROBLEM OF G. E. SACKS

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**Introduction.** On p. 171 of [2] Sacks asks whether there is an r.e. degree of unsolvability $\boldsymbol{d}$ which satisfies $\boldsymbol{O}^{(n)} < \boldsymbol{d}^{(n)} < \boldsymbol{O}^{(n+1)}$ for all $n$. He also conjectures that a proof that such a $\boldsymbol{d}$ exists would use a combinatorial principle not touched on in [2]. In this paper we show using the methods of [2] that such a $\boldsymbol{d}$ exists.[1] Our proof assumes that the reader has a good understanding of [1] and of the proof of Theorem 3 [2, p. 86].

We present our construction in a semi-formal way in the hope that it will thus be made more comprehensible.

1. **Terminology.** By *function* we mean a partial function of natural numbers taking natural number values. We say that the function $\psi$ is an *extension of* the function $\phi$, written $\phi \subseteq \psi$, if the domain of $\psi$ includes that of $\phi$ and if $\phi$ and $\psi$ agree on the domain of $\phi$. We write $\phi = \psi$ just if $\phi \subseteq \psi$ and $\psi \subseteq \phi$; $\phi \neq \psi$ means 'not $\phi = \psi$.' By a *finite function* we mean one whose doman is finite.

Let $\mathfrak{F}$ denote the class of all singulary partial functions with range $\subseteq \{0, 1\}$. By *functional* we mean a map of $\mathfrak{F}$ into itself; we shall denote functionals by capital Greek letters. The image of a function $\phi$ under a functional $\Psi$ is denoted by $\Psi(\phi)$ and the value of $\Psi(\phi)$ for argument $m$ is denoted by $\Psi(\phi; m)$. A functional $\Gamma$ is called *finite* if there is a finite set $\mathfrak{A}$ of ordered pairs of finite functions such that $\Gamma(\eta; x) = y$ if and only if there exists $(\phi, \psi)$ in $\mathfrak{A}$ with $\phi \subseteq \eta$ and $\psi(x) = y$. A sequence $\{\Gamma_i\}$ of finite functionals is called *strongly recursively enumerable* (*s.r.e.*) if given $i$ we can effectively find a finite set $\mathfrak{A}_i$ of ordered pairs of finite functions which specifies $\Gamma_i$ as $\mathfrak{A}$ specifies $\Gamma$ above. Similarly, we define an *s.r.e. double sequence of finite functionals*. A sequence of functionals $\{\Psi_i\}$ is called *increasing* if for every $\eta$ in $\mathfrak{F}$ and every number $i$ we have $\Psi_i(\eta) \subseteq \Psi_{i+1}(\eta)$.

We may define a functional to be *partial recursive* (*p.r.*) if it can be expressed as the limit (in an obvious sense) of an s.r.e. increasing sequence of finite functionals. Below we shall assume the existence of an s.r.e. double sequence $\{\Phi_{i,j}\}$ of finite functionals with the following properties. For fixed $i$ the sequence $\Phi_{i,0}, \Phi_{i,1}, \cdots$ is increasing, with limit $\Phi_i$ say; and $\{\Phi_i\}$ is a standard enumeration of all p.r. functionals.

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[1] Mr. D. A. Martin has independently obtained an affirmative answer to (Q4) on p. 171 of [2] by an argument different from the one in this paper.

By the characteristic function of a set of natural numbers we mean the total function which is 0 for arguments in the set and 1 otherwise. We find it convenient to identify sets with their characteristic functions.

Let $S$ be any set, then the sequence of sets

$$\{x \mid \Phi_i(S; x) \text{ is defined}\}, \qquad i = 0, 1, \cdots, \tag{1}$$

is a standard enumeration of the sets recursively enumerable in $S$, and the set

$$\{x \mid \Phi_x(S; x) \text{ is defined}\}$$

denoted by $S'$ is (1-1)-complete for the sets (1) uniformly in $S$. Also, for any set $S$ we let $S^{(0)} = S$ and $S^{(i+1)} = S^{(i)\prime}$ for all $i$. If $S$ has degree $\mathbf{s}$, then $S'$ has degree $\mathbf{s}'$.

The empty set is denoted by $\varnothing$, and when we take $S = \varnothing^{(j)}$ the sets (1) are called *j-enumerable*; we adopt the indexing of these sets provided by (1). A set $T$ is called *j-recursive* if both $T$ and its complement $\overline{T}$ are $j$-enumerable; such a set is to be specified by indices of $T$ and $\overline{T}$ as $j$-enumerable sets.

Below we shall require the following elementary result: given a finite functional $\Gamma$, a $j$-recursive set $S$, and numbers $m$, $n$ such that $\Gamma(S; m) = n$, we can effectively find assuming a knowledge of $\varnothing^{(j)}$ a number $p$ such that, if $T$ is any set which differs from $S$ only as regards members $\geqq p$, then $\Gamma(T; m) = n$ also. We say that the number $p$ *fixes* the value $\Gamma(S; m) = n$.

For any two sets $S$, $T$ we let $S \oplus T$ denote the set $\{2x \mid x \in S\} \cup \{2x+1 \mid x \in T\}$.

2. **Preliminaries.** By a close inspection of the proof of Sacks's Theorem 3 [2, p. 86], we obtain:

LEMMA 1. *There are recursive functions $\alpha$, $\beta$ and a p.r. functional $\Theta$ such that, if $e$ is the index of a j-enumerable set $C \oplus \varnothing^{(j)}$ for any $j > 0$, then $\alpha(e)$ is the index of a $(j-1)$-enumerable set $D \oplus \varnothing^{(j-1)}$ such that*

$$\Theta((D \oplus \varnothing^{(j-1)})') = C \oplus \varnothing^{(j)},$$

*and*

$$\Phi_{\beta(e)}(C \oplus \varnothing^{(j)}) = (D \oplus \varnothing^{(j-1)})'.$$

Of course, Sacks's construction needs a little pulling and pushing to give us this lemma. Roughly speaking, our lemma corresponds to the case of his theorem in which $\boldsymbol{b} = \boldsymbol{g} = \boldsymbol{O}^{(j-1)}$ and in which $\boldsymbol{c}$, $\boldsymbol{d}$ are the degrees of $C \oplus \varnothing^{(j)}$, $D \oplus \varnothing^{(j-1)}$ respectively. The $\boldsymbol{a}$ of Sacks's theorem is to be disregarded and the provisions made for it are to be excised. To give more than this meagre explanation would entail repeating a lot of Sacks's proof in a simpler situation.

LEMMA 2. *There is a recursive function $\gamma$ with the following property. If $S^*$, $T^*$ are sets agreeing with $S$, $T$ respectively for members $\geqq x$ and if $\Phi_i(S^*) = T^*$, then $\Phi_j(S) = T$ for some $j$, $j \leqq \gamma(i, x)$.*

PROOF. There are p.r. functionals $\Psi_0^x, \Psi_1^x, \cdots, \Psi_{2^x-1}^x$ such that, for any set $X$, $\Psi_i^x(X)$ with $x$ fixed runs through all sets (meaning all characteristic functions thereof) which agree with $X$ for members $\geqq x$. Hence from $\Phi_i(S^*) = T^*$ we can infer $\Psi_m^x \Phi_i \Psi_n^x(S) = T$ for some $m$ and $n$ both $< 2^x$. But, given $x$, we can effectively construct $\Psi_0^x, \Psi_1^x, \cdots, \Psi_{2^x-1}^x$ and so we can effectively find in terms of $i$ and $x$ a bound for the least index of $\Psi_m^x \Phi_i \Psi_n^x$.

LEMMA 3. *There is a recursive function $\gamma$ such that for any sets $S$, $T$, $\Phi_i(S) = T$ implies $\Phi_{\gamma(i)}(S') = T'$.*

PROOF. By definition we have

$$T' = \{x \mid \Phi_x(T; x) \text{ is defined}\},$$

and the r.h.s. is just $\{x \mid \Phi_x \Phi_i(S; x) \text{ is defined}\}$. We can define a binary recursive function $\alpha$ so that the last set is $\{x \mid \Phi_{\alpha(i,x)}(S; \alpha(i, x)) \text{ is defined}\}$. Thus $\alpha(i, x)$ considered as a function of $x$ constitutes a many-one reduction of $T'$ to $S'$. We can effectively find the index $\gamma(i)$ of the p.r. functional which maps $\phi$ into the function $\psi$ defined by

$$\psi(x) = \phi(\alpha(i, x)).$$

It is clear that $\Phi_{\gamma(i)}(S') = T'$ as required.

The next lemma combines Lemmas 2 and 3 in a result which we shall use below.

LEMMA 4. *There is a ternary recursive function $\gamma$ with the following property. Suppose for $k = 0, 1$ that $C^k \oplus \varnothing^{(j)}$ is the $j$-enumerable set with index $e^k$ and that $D^k \oplus \varnothing^{(j-1)}$ is the $(j-1)$-enumerable set with index $\alpha(e^k)$. Suppose further that $E^k$ agrees with $D^k \oplus \varnothing^{(j-1)}$ for members $\geqq x$, and that $\Phi_i(E^0) = E^1$. Then for some $p$, $p \leqq \gamma(e^0, x, i)$, we have $\Phi_p(C^0 \oplus \varnothing^{(j)}) = C^1 \oplus \varnothing^{(j)}$.*

PROOF. From Lemma 2 we have

$$\Phi_m(D^0 \oplus \varnothing^{(j-1)}) = D^1 \oplus \varnothing^{(j-1)}$$

for some $m$, $m \leqq \gamma(i, x)$. Further, from Lemma 3

$$\Phi_{\gamma(m)}((D^0 \oplus \varnothing^{(j-1)})') = (D^1 \oplus \varnothing^{(j-1)})'.$$

Using Lemma 1 it follows that

$$\Theta\Phi_{\gamma(m)}\Phi_{\beta(e^0)}(C^0 \oplus \varnothing^{(j)}) = C^1 \oplus \varnothing^{(j)}.$$

The required ternary function $\gamma$ can now be defined easily.

We can suppose without loss of generality that $\gamma$ is increasing in all its arguments and also that $\gamma(e, x, i) \geqq i$; these assumptions will be used below.

The singulary function $\alpha$ and the ternary function $\gamma$ both play an important role in the construction to follow.

3. **The construction.** The idea of the construction is as follows. We construct two sequences $A^{0,0}, A^{0,1}, \cdots; A^{1,0}, A^{1,1}, \cdots$ of sets so that $A^{0,j}$ and $A^{1,j}$ are $j$-enumerable. Sacks's construction is used to ensure that $A^{k,j+1}$ has the same degree as $(A^{k,j})'$. Also, we use the method of Friedberg [1] in constructing $A^{0,j}$ and $A^{1,j}$ to ensure that

$$\Phi_x(A^{0,y}) \neq A^{1,y} \quad \text{and} \quad \Phi_x(A^{1,y}) \neq A^{0,y}$$

for all $x \leqq j$ and $y \leqq j$. It is clear that, if we can realise these aims, then the problem posed by Sacks is solved.

However, because we have $\Phi_{\beta(e)}$ rather than a fixed p.r. functional in Lemma 1, we have to proceed in a rather roundabout way. We suppose that we are given (by index) two p.r. functions $\phi^0, \phi^1$. From these we construct simultaneously two sequences of sets $A^{0,0}, A^{0,1}, \cdots$ and $A^{1,0}, A^{1,1}, \cdots$. We shall arrange later for $\phi^0, \phi^1$ to be recursive and for $A^{k,j+1}$ to be the $(j+1)$-enumerable set with index $\phi^k(j)$.

The construction consists of an infinite number of Steps which we number $0, 1, 2, \cdots$. We denote by $A_n^{k,j}$ the set of numbers enumerated in $A^{k,j}$ in the Steps $\leqq n$. At Step $n$ we shall (among other things) define the auxiliary functions $\delta_n^{k,j}, \lambda_n, \pi_n$ for some arguments; in fact, $\lambda_n(x)$ is defined for all $x$ and $\pi_n(x)$ is defined for $x \leqq n$.

*Step* 0. We set $A_0^{0,j} = A_0^{1,j} = \varnothing \oplus \varnothing^{(j)}$ and $\lambda_0(j) = 0$ for all $j$. We set $\pi_0(0) = 0$.

*Step* $n+1$. We first compute $\phi^0(n)$ and $\phi^1(n)$. If either of these is undefined, the construction proceeds no further.

We then carry out the following $2n+2$ Substeps which are numbered $0, 1, \cdots, 2n+1$. For $n$ even we prescribe:

*Substep* $2j$. We seek the least number $e$, $e \leqq \pi_n(j)$, such that either

$$\Phi_{e,n}(A_n^{0,j}; \delta_n^{0,j}(e)) = A_n^{1,j}(\delta_n^{0,j}(e)) = 1, \tag{2}$$

or $\delta_n^{1,j}(e)$ is undefined. There are three cases.

*Case* 0. If (2) holds, let $p$ be a number, the least say, which fixes the value $\Phi_{e,n}(A_n^{0,j};\ \delta_n^{0,j}(e))$. We define $\lambda_{n+1}(j)$ to be $\mathrm{Max}\{p,\ \lambda_n(j)\}$; we enumerate $\delta_n^{0,j}(e)$ in $A^{1,j}$; also, we define

$$\delta_{n+1}^{0,j}(x) = \delta_n^{0,j}(x),$$

$$\delta_{n+1}^{1,j}(x) = \begin{cases} \delta_n^{1,j}(x) & \text{if } x < e, \\ \text{undefined} & \text{if } x \geqq e. \end{cases}$$

*Case* 1. If (2) does not hold and $\delta_n^{1,j}(e)$ is not defined, then we define $\delta_{n+1}^{1,j}(e)$ to be the least even number $\geqq \lambda_n(j)$ not in $A_n^{0,j}$, and we define $\lambda_{n+1}(j)$ to be $\delta_{n+1}^{1,j}(e)+1$. Also, $\delta_{n+1}^{0,j}$ and $\delta_{n+1}^{1,j}$ are to be the same $\delta_n^{0,j}$ and $\delta_n^{1,j}$ respectively, except as just provided for $\delta_{n+1}^{n}(e)$.

*Case* 2. Otherwise, that is, if there is no $e \leqq \pi_n(j)$ such that (2) holds or such that $\delta_n^{1,j}(e)$ is undefined. Then $\delta_{n+1}^{0,j}$, $\delta_{n+1}^{1,j}$ are to be the same as $\delta_n^{0,j}$, $\delta_n^{1,j}$ respectively.

*Substep* $2j+1$. We enumerate in $A^{0,j}$ all the even numbers in the finite set

$$\{x \mid \Phi_{\alpha(\phi^0(j)),n}(\varnothing^{(j)};\ x) \text{ is defined}\}$$

which are $\geqq \lambda_{n+1}(j)$. ($\lambda_{n+1}(j)$ is $\lambda_n(j)$ if not otherwise defined.)

If $n$ is odd, then Substep $2j$ is to be obtained from the one given by interchanging the superscripts 0, 1 and by interchanging the Cases 0, 1. Thus Case 0 will now read: "If $\delta_n^{0,j}(e)$ is not defined, then . . . ." Also, in the new Case 1 we must write $x \leqq e$, $x > e$ for $x < e$, $x \geqq e$ respectively. In Substep $2j+1$ we read $A^{1,j}$ for $A^{0,j}$, and $\phi^1(j)$ for $\phi^0(j)$.

To complete Step $n+1$ we define $\lambda_{n+1}(x)$ to be $\lambda_n(x)$ if $\lambda_{n+1}(x)$ has not yet been defined, and we define $\pi_{n+1}(x)$ for $x \leqq n+1$ as follows.

$$\begin{aligned}
&\pi_{n+1}(0) = 0,\\
&\pi_{n+1}(1) = \gamma(\mathrm{Max}\{\phi^0(0), \phi^1(0)\}, \lambda_{n+1}(0), 1),\\
&\pi_{n+1}(2) = \gamma(\mathrm{Max}\{\phi^0(1), \phi^1(1)\}, \lambda_{n+1}(1),\\
&\qquad\qquad \gamma(\mathrm{Max}\{\phi^0(0), \phi^1(0)\}, \lambda_{n+1}(0), 2)),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad (3)\\
&\pi_{n+1}(n+1) = \gamma(\mathrm{Max}\{\phi^0(n), \phi^1(n)\}, \lambda_{n+1}(n),\\
&\qquad\qquad \gamma(\mathrm{Max}\{\phi^0(n-1), \phi^1(n-1)\}, \lambda_{n+1}(n-1), \cdots,\\
&\qquad\qquad \gamma(\mathrm{Max}\{\phi^0(0), \phi^1(0)\}, \lambda_{n+1}(0), n+1) \cdots)).
\end{aligned}$$

This completes the construction, which can only terminate after a finite number of Steps by $\phi^0$ or $\phi^1$ being undefined for some argument. If all the Steps $\leqq n$ terminate, then $A_n^{0,j}$, $A_n^{1,j}$ are $j$-recursive sets.

Further, their $j$-recursiveness is uniform in $j$ and $n$ in the sense that given $j$ and $n$ we can effectively find indices of $A_n^{0,j}$, $A_n^{1,j}$ and their complements as $j$-enumerable sets. Also, for $k=0$, 1 and all $j$ we have $A_n^{k,j}$ increasing to the limit $A^{k,j}$ as $n$ increases. This means that $A^{k,j}$ is a $j$-enumerable set. Further, there are recursive functions $\psi^0$, $\psi^1$ such that for all $j$ and $k=0$, 1 we have $\psi^k(j)$ is the index of $A^{k,j+1}$ as a $(j+1)$-enumerable set. Further, indices of $\psi^0$ and $\psi^1$ can be found effectively given indices of $\phi^0$ and $\phi^1$. It now follows by the second recursion theorem that we may choose $\phi^0$, $\phi^1$ so that $\phi^0=\psi^0$ and $\phi^1=\psi^1$. In this case $\phi^0$, $\phi^1$ will be recursive, because $\psi^0$, $\psi^1$ are. Below we shall suppose that $\phi^0$ and $\phi^1$ have been so chosen. Thus for $k=0$, 1 and all $j$, $\phi^k(j)$ is the index of $A^{k,j+1}$ as a $(j+1)$-enumerable set.

To show that the construction has the hoped-for effect, we first prove:

LEMMA 5. *For all* $j$
(a) $\pi_n(j)$ *has a limit* $\pi(j)$ *as* $n$ *increases*;
(b) *if* $x \leqq \pi(j)$, *then* $\Phi_x(A^{0,j}) \neq A^{1,j}$ *and* $\Phi_x(A^{1,j}) \neq A^{0,j}$;
(c) $\lambda_n(j)$ *has a limit* $\lambda(j)$ *as* $n$ *increases.*

PROOF. We use course-of-values induction on $j$. For $j=0$, (a) is immediate.

Assuming that (a) holds for some $j$ we can prove (b) and (c) for the same $j$. We need only consider the Substeps $2j$, $2j+1$ at Steps $n+1$ such that $\pi_x(j)=\pi(j)$ for all $x \geqq n$. Now Substep $2j$ is modelled on the construction [1] except that where Friedberg was concerned with all the p.r. functionals we are concerned only with the initial segment $\{\Phi_x \mid x \leqq \pi(j)\}$. In adapting Friedberg's proof we would prove the propositions:

(4, x) $\delta_n^{0,j}(x)$ has a limit $\delta^{0,j}(x)$ such that
$$\Phi_x(A^{0,j};\delta^{0,j}(x)) \neq A^{1,j}(\delta^{0,j}(x)),$$

(5, x) $\delta_n^{1,j}(x)$ has a limit $\delta^{1,j}(x)$ such that
$$\Phi_x(A^{1,j};\delta^{1,j}(x)) \neq A^{0,j}(\delta^{1,j}(x))$$

for $x \leqq \pi(j)$ in the order (4, 0), (5, 0), (4, 1), (5, 1), $\cdots$, (4, $\pi(j)$), (5, $\pi(j)$).

After $\pi_n(j)$, and $\delta_n^{0,j}(x)$, $\delta_n^{1,j}(x)$ for $x \leqq \pi(j)$, have all reached their limits, Case 0 or 1 can only apply in Substep $2j$ of one further Step. Thus for all but a finite number of Steps Case 2 obtains in Substep $2j$. Hence $\lambda_n(j)$ has a limit $\lambda(j)$. We should have observed already that Substep $2j+1$ is specifically designed so as not to conflict with the purpose of Substep $2j$.

To complete the proof of the lemma we need only remark from the equations (3) that, if $\lambda_n(0), \lambda_n(1), \cdots, \lambda_n(j)$ have the respective limits $\lambda(0), \lambda(1), \cdots, \lambda(j)$, then $\pi_n(j+1)$ has the limit

$$\gamma(\mathrm{Max}\{\phi^0(j), \phi^1(j)\}, \lambda(j), \gamma(\mathrm{Max}\{\phi^0(j-1), \phi^1(j-1)\}, \lambda(j-1), \cdots, \gamma(\mathrm{Max}\{\phi^0(0), \phi^1(0)\}, \lambda(0), j+1) \cdots)).$$

LEMMA 6. *For all $j$ and $k=0, 1$, $A^{k,j+1}$ is a $(j+1)$-enumerable set of the form $X \oplus \varnothing^{(j+1)}$ with index $\phi^k(j)$, and $A^{k,j}$ agrees with the $j$-enumerable set with index $\alpha(\phi^k(j))$ for members $\geqq \lambda(j)$.*

PROOF. After Step 0 only even numbers can enter $A^{k,j}$, and so by Step 0 we know that $A^{k,j}$ is of the form $X \oplus \varnothing^{(j)}$. We arranged above for $A^{k,j+1}$ to have index $\phi^k(j)$ and so the $j$-enumerable set with index $\alpha(\phi^k(j))$, call it $W$, has the form $X \oplus \varnothing^{(j)}$ by Lemma 1. Thus $A^{k,j}$ and $W$ certainly agree for odd members. Now $\lambda(j)$ is the limit of $\lambda_n(j)$ and the set

$$\{x \mid \Phi_{\alpha(\phi^k(j)),n}(\varnothing^{(j)}; x) \text{ is defined}\}$$

increases with $n$ to the limit $W$. Hence by the Substeps $2j+1$ we have $A^{k,j}$ agreeing with $W$ for members $\geqq \lambda(j)$.

It is clear from Lemmas 1 and 6 that the degree of $A^{k,j+1}$ is the jump of the degree of $A^{k,j}$. To complete the proof it suffices to show that

$$\Phi_x(A^{0,j}) \neq A^{1,j} \quad \text{and} \quad \Phi_x(A^{1,j}) \neq A^{0,j} \tag{6}$$

for all $x$ and $j$. When $x \leqq j$, (6) follows by (b) of Lemma 5, because $\gamma(e, x, i) \geqq i$ which means that $\pi(j) \geqq j$. Now consider the case $x=m$, $j=n$ with $m>n$. Then by Lemmas 4 and 6 we have that

$$\Phi_m(A^{0,n}) \neq A^{1,n} \quad \text{and} \quad \Phi_m(A^{1,n}) \neq A^{0,n} \tag{7}$$

holds provided that (6) holds for $j=n+1$ and

$$x \leqq \gamma(\mathrm{Max}\{\phi^0(n), \phi^1(n)\}, \lambda(n), m).$$

However, by another application of Lemma 4 we see that this last condition holds provided that (6) holds for $j=n+2$ and

$$x \leqq \gamma(\mathrm{Max}\{\phi^0(n+1), \phi^1(n+1)\}, \lambda(n+1), \gamma(\mathrm{Max}\{\phi^0(n), \phi^1(n)\}, \lambda(n), m)).$$

And, repeating the application of Lemma 4 a finite number of times, we can see that (7) will hold provided that (6) holds for $j=m$ and

$$x \leqq \gamma(\mathrm{Max}\{\phi^0(m-1), \phi^1(m-1)\}, \lambda(m-1), \gamma(\mathrm{Max}\{\phi^0(m-2), \phi^1(m-2)\}, \lambda(m-2), \cdots, \gamma(\mathrm{Max}\{\phi^0(n), \phi^1(n)\}, \lambda(n), m) \cdots)).$$

By inspection the r.h.s. of this inequality is $\leqq \pi(m)$ and so (7) follows by (b) of Lemma 5 with $j=m$. This completes the proof of (6).

Taking $\boldsymbol{a}_0$, $\boldsymbol{a}_1$ to be the degrees of $A^{0,0}$, $A^{1,0}$ respectively we have shown that $\boldsymbol{a}_0^{(n)}$ and $\boldsymbol{a}_1^{(n)}$ are incomparable for all $n$. We may take either $\boldsymbol{a}_0$ or $\boldsymbol{a}_1$ to be the $\boldsymbol{d}$ whose existence was queried by Sacks.

## References

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