NOTE ON A PROPERTY OF THE ELEMENTARY SYMMETRIC FUNCTIONS

J. N. DARROCH AND JANE PITMAN

Let \( r_i \) \((i = 1, 2, \ldots, n)\) be \( n \) real numbers, and consider the polynomial

\[
G(x) = (1 + r_1x)(1 + r_2x) \cdots (1 + r_nx)
\]

where \( c_0 = 1 \) and, for \( k = 1, 2, \ldots, n \), \( c_k \) is the \( k \)th elementary symmetric function of the \( r_i \).

It follows from Newton's inequality (given as (6) in Chapter 1, \( \S 12 \), of [1]) that, for \( k = 1, 2, \ldots, n-1 \),

\[
(1) \quad c_{k-1}c_{k+1} \leq c_k^2
\]

with strict inequality unless \( c_k = 0 \). We use this to prove the following two theorems.

**Theorem 1.** If the \( r_i \) \((i = 1, 2, \ldots, n)\) are all positive, then, for \( k = 1, 2, \ldots, n-1 \),

\[
(2) \quad c_{k+1} \leq c_k \Rightarrow c_k > c_{k-1}.
\]

**Proof.** If the \( r_i \) are all positive, then the \( c_k \) are all positive, so that (1) holds in its strict form, and

\[
\frac{c_{k-1}}{c_k} \leq \frac{c_k}{c_{k+1}} \quad (k = 1, 2, \ldots, n - 1),
\]

from which (2) follows.

**Theorem 2.** If the \( r_i \) are all positive, then, for \( k = 2, 3, \ldots, n-1 \),

(3) \( (c_{k+1} - 2c_k + c_{k-1} \geq 0) \land (c_k > c_{k-1}) \Rightarrow c_k - 2c_{k-1} + c_{k-2} > 0 \),

(4) \( (c_k - 2c_{k-1} + c_{k-2} \geq 0) \land (c_k < c_{k-1}) \Rightarrow c_{k+1} - 2c_k + c_{k-1} > 0 \).

**Proof of (3).** Suppose the \( r_i \) are all positive and \( c_k > c_{k-1} \). Consider the polynomial

\[
G(x)(1 - x) = c_0 + (c_1 - c_0)x + (c_2 - c_1)x^2 + \cdots + (c_n - c_{n-1})x^n - c_nx^{n+1}.
\]

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Since $c_k - c_{k-1} \neq 0$, we may apply (1) in its strict form to the coefficients of $x^{k-1}, x^k, x^{k+1}$ in this polynomial, and so obtain

$$(c_{k-1} - c_{k-2})(c_{k+1} - c_k) < (c_k - c_{k-1})^2.$$ 

By Theorem 1, $c_{k-1} > c_{k-2}$, and therefore

$$(5) \quad \frac{c_{k+1} - c_k}{c_k - c_{k-1}} < \frac{c_k - c_{k-1}}{c_{k-1} - c_{k-2}}.$$ 

If, further, $c_{k+1} - 2c_k + c_{k-1} \geq 0$, then

$$1 \leq \frac{c_{k+1} - c_k}{c_k - c_{k-1}},$$

and so we deduce from (5) that $c_k - 2c_{k-1} + c_{k-2} > 0$.

(The proof of (4) is similar.)

These two theorems describe the shape of the coefficient function $f$ defined by $f(k) = c_k$ ($k = 0, 1, 2, \cdots, n$). Theorem 1 shows that $f$ is either first increasing and then decreasing or always increasing or always decreasing, and that $f$ attains its maximum at most twice. Theorem 2 shows that if $f$ is increasing and convex at some point then it is convex to the left of that point, and that if $f$ is decreasing and convex at some point then it is convex to the right of that point. We may combine these conclusions by saying that $f$ has either a bell-shaped graph or a truncated bell-shaped graph.

In [2], Darroch stated these theorems in a probability context and proved them by induction using probability arguments. His Theorems 1 and 2 may be obtained by applying Theorems 1 and 2 above to the polynomial $G(x)$ defined by

$$q_1q_2\cdots q_n G(x) = (q_1 + p_1x)(q_2 + p_2x)\cdots(q_n + p_nx),$$

where $p_i = 1 - q_i$ is the probability of success in the $i$th trial of a sequence of $n$ independent Bernoulli trials. Darroch also proved a theorem concerning the location of the maximum of the coefficient function for this polynomial.

References


University of Adelaide, Adelaide, Australia