NOTE ON A PROPERTY OF THE ELEMENTARY SYMMETRIC FUNCTIONS

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Let r_i $(i=1, 2, \cdots, n)$ be n real numbers, and consider the polynomial

$$G(x) = (1 + r_1x)(1 + r_2x) \cdots (1 + r_nx)$$
$$= c_0 + c_1x + c_2x^2 + \cdots + c_nx^n,$$

where $c_0=1$ and, for $k=1, 2, \cdots, n$, c_k is the kth elementary symmetric function of the r_i.

It follows from Newton's inequality (given as (6) in Chapter 1, §12, of [1]) that, for $k=1, 2, \cdots, n-1$,

$$c_{k-1}c_{k+1} \leqq c_k^2, \tag{1}$$

with strict inequality unless $c_k=0$. We use this to prove the following two theorems.

THEOREM 1. *If the* r_i $(i=1, 2, \cdots, n)$ *are all positive, then, for* $k=1, 2, \cdots, n-1$,

$$c_{k+1} \geqq c_k \Rightarrow c_k > c_{k-1}. \tag{2}$$

PROOF. If the r_i are all positive, then the c_k are all positive, so that (1) holds in its strict form, and

$$\frac{c_{k-1}}{c_k} < \frac{c_k}{c_{k+1}} \qquad (k = 1, 2, \cdots, n-1),$$

from which (2) follows.

THEOREM 2. *If the* r_i *are all positive, then, for* $k=2, 3, \cdots, n-1$,

$$(c_{k+1} - 2c_k + c_{k-1} \geqq 0) \wedge (c_k > c_{k-1}) \Rightarrow c_k - 2c_{k-1} + c_{k-2} > 0, \tag{3}$$

$$(c_k - 2c_{k-1} + c_{k-2} \geqq 0) \wedge (c_k < c_{k-1}) \Rightarrow c_{k+1} - 2c_k + c_{k-1} > 0. \tag{4}$$

PROOF OF (3). Suppose the r_i are all positive and $c_k>c_{k-1}$. Consider the polynomial

$$G(x)(1 - x) = c_0 + (c_1 - c_0)x + (c_2 - c_1)x^2 + \cdots$$
$$+ (c_n - c_{n-1})x^n - c_nx^{n+1}.$$

Received by the editors November 16, 1964.

Since $c_k - c_{k-1} \neq 0$, we may apply (1) in its strict form to the coefficients of x^{k-1}, x^k, x^{k+1} in this polynomial, and so obtain

$$(c_{k-1} - c_{k-2})(c_{k+1} - c_k) < (c_k - c_{k-1})^2.$$

By Theorem 1, $c_{k-1} > c_{k-2}$, and therefore

$$\frac{c_{k+1} - c_k}{c_k - c_{k-1}} < \frac{c_k - c_{k-1}}{c_{k-1} - c_{k-2}}. \tag{5}$$

If, further, $c_{k+1} - 2c_k + c_{k-1} \geqq 0$, then

$$1 \leqq \frac{c_{k+1} - c_k}{c_k - c_{k-1}},$$

and so we deduce from (5) that $c_k - 2c_{k-1} + c_{k-2} > 0$.

(The proof of (4) is similar.)

These two theorems describe the shape of the coefficient function f defined by $f(k) = c_k$ ($k = 0, 1, 2, \cdots, n$). Theorem 1 shows that f is *either* first increasing and then decreasing *or* always increasing or always decreasing, and that f attains its maximum at most twice. Theorem 2 shows that if f is increasing and convex at some point then it is convex to the left of that point, and that if f is decreasing and convex at some point then it is convex to the right of that point. We may combine these conclusions by saying that f has either a bell-shaped graph or a truncated bell-shaped graph.

In [2], Darroch stated these theorems in a probability context and proved them by induction using probability arguments. His Theorems 1 and 2 may be obtained by applying Theorems 1 and 2 above to the polynomial $G(x)$ defined by

$$q_1 q_2 \cdots q_n G(x) = (q_1 + p_1 x)(q_2 + p_2 x) \cdots (q_n + p_n x),$$

where $p_i = 1 - q_i$ is the probability of success in the ith trial of a sequence of n independent Bernoulli trials. Darroch also proved a theorem concerning the location of the maximum of the coefficient function for this polynomial.

References

1. E. F. Beckenbach and R. Bellman, *Inequalities*, Springer, Berlin, 1961.

2. J. N. Darroch, *On the distribution of the number of successes in independent trials*, Ann. Math. Statist. 35 (1964), 1317–1321.

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