ON THE INVARIANTS OF A VECTOR SUBSPACE OF A VECTOR SPACE OVER A FIELD OF CHARACTERISTIC TWO

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1. **Introduction.** Witt's theorem is concerned with the extension of an isometry between subspaces to an isometry on the whole space. The most general form of Witt's theorem is Theorem 1.2.1 in Wall [**3**]. Theorem 1 of this paper extends Theorem 1.2.1 and is identical to it in case the characteristic of the division ring is not 2. Theorem 2 is a variant of Theorem 1. Theorems 1 and 2 are concerned with sesquilinear forms. Theorems 3 and 4 are concerned with bilinear forms on a finite dimensional vector space over a field of characteristic 2. Theorem 3 gives necessary and sufficient conditions for two (possibly degenerate) forms to be equivalent. Theorem 4 gives necessary and sufficient conditions for two subspaces to be equivalent.

The original results of this paper were based on results in Dieudonné [**1**]. However, the referee kindly pointed out that the proofs can be simplified and some of the results generalized by using results in Wall [**3**]. In particular he pointed out that Wall's proof is valid for the results stated in Theorem 1 as the restrictions contained in Theorem 1.2.1, are not necessary. He also suggested the variant on Theorem 1 which is Theorem 2. The proof of Theorem 4 has been considerably simplified by the use of Theorem 2. I wish to thank the referee for these suggestions as it allows me to present these results in a more elegant and simplified form.

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2. **Notation.** Let V be a vector space of possibly infinite dimension over a division ring D with a fixed involutory anti-automorphism J, that is, a one-to-one mapping $\alpha \to \alpha^J$ of D onto itself such that $(\alpha+\beta)^J = \alpha^J + \beta^J$, $(\alpha\beta)^J = \beta^J\alpha^J$, and $\alpha^{J^2} = \alpha$. An Hermitian (skew-Hermitian) sesquilinear form on V is a mapping $f\colon V \times V \to D$ such that $f(x, y)$ is linear in x for each fixed y and $f(y, x) = f(x, y)^J$ ($f(y, x) = -f(x, y)^J$). If the characteristic of D is two, the distinction between Hermitian and skew-Hermitian forms disappears.

Two forms f_1 and f_2 are called equivalent if there is a linear trans-

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formation σ of V_1 onto V_2 with the property that

$$f_1(x, y) = f_2(\sigma(x), \sigma(y))$$

for all x and y in V_1; and σ is called an isometry.

If $W \subset V$, $W^\perp$ is the set of all y in V such that $f(x, y) = 0$ for all x in W.

A form f is called nondegenerate if $V^\perp = 0$. Otherwise f is called degenerate.

3. **Witt's theorem.** Let f be a nondegenerate Hermitian or skew-Hermitian form on V. The set of all isometries of V onto itself form the unitary group $U(f)$. If σ is in $U(f)$ and the range of $I - \sigma$ is finite dimensional, then σ is said to be finite dimensional. The finite dimensional elements of $U(f)$ form a normal subgroup of $U_\phi(f)$.

If x is in V, we call x trace-valued if $f(x, x) = \lambda + \epsilon\lambda^J$, $\epsilon = \pm 1$, for some λ in D. By Lemma 1.2.1 in [**3**] the set of trace-valued vectors in V forms a subspace V^τ of V. If the characteristic of D is not 2, $V^\tau = V$.

The most general form of Witt's theorem is Wall's Theorem 1.2.1 [**3**] which we now state.

THEOREM 1.2.1 (WITT). *Let W_1, W_2 be finite dimensional subspaces of V^τ such that $W_1 \cap (V^\tau)^\perp = W_2 \cap (V^\tau)^\perp = \{0\}$. Then every isometry σ of W_1 onto W_2 can be extended to an element of $U_\phi(f)$.*

The following is an extension of the preceding theorem and a generalization of the characteristic two case in [**2**].

THEOREM 1 (WITT). *Let W_1, W_2 be finite dimensional subspaces of V such that $W_1 \cap (V^\tau)^\perp = W_2 \cap (V^\tau)^\perp$. Then every isometry of W_1 onto W_2 which is the identity on $W_1 \cap (V^\tau)^\perp$ can be extended to an element of $U_\phi(f)$.*

PROOF. By Lemma 1.2.2, Corollary [**3**], every element in $U(f)$ leaves $(V^\tau)^\perp$ pointwise invariant. Hence it is enough to be able to extend the isometry to an element of $U_\phi(f)$ under the assumption that $W_1 \cap (V^\tau)^\perp = W_2 \cap (V^\tau)^\perp = \{0\}$. This can be proved by following Wall's proof of Theorem 1.2.1 and deleting the restriction that $W_1, W_2 \subset V^\tau$. This restriction is nowhere needed in the proof. Q.E.D.

If W_1 and W_2 are subspaces of V and W_1 is isometric to W_2 by an isometry in $U_\phi(f)$ we will call W_1 and W_2 equivalent. Theorem 2 is a variant of Theorem 1.

THEOREM 2. *If W_1 and W_2 are finite dimensional subspaces of V, then W_1 is equivalent to W_2 if, and only if,* (1) $W_1 \cap (V^\tau)^\perp = W_2 \cap (V^\tau)^\perp$ *and* (2) *W_1 is isometric to W_2.*

PROOF. The first condition is necessary by Lemma 1.2.2, Corollary in [3]. To prove the sufficiency we need only show that there is an isometry ω sending W_1 onto W_2 which sends $W_1 \cap (V^\tau)^\perp$ onto $W_2 \cap (V^\tau)^\perp$. Then ω would send a complement of $W_1 \cap (V^\tau)^\perp$ in W_1 onto a complement of $W_2 \cap (V^\tau)^\perp$ in W_2. Hence by the proof of Theorem 1, there exists a σ in $U_\phi(f)$ sending W_1 onto W_2.

To establish the existence of ω, let ρ be an isometry of W_1 onto W_2 such that the subspace $X = \{x \mid x \text{ is in } W_1 \cap (V^\tau)^\perp \text{ and } \rho(x) \text{ is in } W_2 \cap (V^\tau)^\perp\}$ have as large a dimension as possible. We assert X equals $W_1 \cap (V^\tau)^\perp$. We will prove this by contradiction. Suppose that X does not equal $W_1 \cap (V^\tau)^\perp$. Then there is an a in $W_1 \cap (V^\tau)^\perp$ with $\rho(a)$ not in $W_2 \cap (V^\tau)^\perp$. Hence it is possible to find an a such that in addition to $a \in W_1 \cap (V^\tau)^\perp$, $\rho(a) \notin W_2 \cap (V^\tau)^\perp$, we have $a = \rho(b)$ where $b \in W_1$, $b \notin W_1 \cap (V^\tau)^\perp$. Let Y' be the subspace generated by a and b. Then dim $Y' = 2$ and $X \cap Y' = 0$. Hence we can choose a complement Y of Y' in W_1 such that $X \subset Y$.

Now we define a new isometry ψ of W_1 onto W_2 as follows. $\psi(x) = \rho(x)$ for x in Y. $\psi(a) = a$. $\psi(b) = \rho(a)$. If ψ is indeed an isometry we will have our contradiction. Clearly ψ is one-to-one so we have to verify that $f(\psi(x), a) = f(x, a)$ and $f(\psi(x), \rho(a)) = f(x, b)$ for all x in W_1. By Lemma 1.2.2, Corollary [3], $\rho(x) - x$ is in V^τ for all x in V so that $f(\rho(x), a) = f(x, a)$ for all x in W_1. If x is in Y, $f(\psi(x), a) = f(\rho(x), a) = f(x, a)$. It can be shown that $f(\psi(b), a) = f(b, a)$ also, so that $f(\psi(x), a) = f(x, a)$ for all x in W_1. For x in Y, $f(\psi(x), \rho(a)) = f(\rho(x), \rho(a)) = f(x, a) = f(\rho(x), a) = f(\rho(x), \rho(b)) = f(x, b)$. It can also be shown that $f(\psi(a), \rho(a)) = f(a, b)$, so that $f(\psi(x), \rho(a)) = f(x, b)$ for all x in W_1. Q.E.D.

A subspace $W \subset V$ is called isotropic if $f(x, y) = 0$ for all x and y in W.

COROLLARY 2.1. *If W_1 and W_2 are two isotropic subspaces of V such that $W_1 \cap (V^\tau)^\perp = W_2 \cap (V^\tau)^\perp$, W_1 and W_2 are equivalent.*

4. **Invariants.** In this section we assume that V is finite dimensional, J is the identity, and D is a field of characteristic two. Under these assumptions f is a nondegenerate symmetric bilinear form, and V^τ is the set of all x such that $f(x, x) = 0$. Let $m = \dim V$.

Since $\lambda \rightarrow \lambda^2$ is an automorphism of D into D^2, the mapping $\theta: x \rightarrow f(x, x)$ is a semi-linear transformation of V into the vector space D over the field D^2. Let $\boldsymbol{W} = \theta(V)$. Clearly $\boldsymbol{W}$ is a subspace of D over D^2. V^τ equals $\theta^{-1}(0)$. Let $\boldsymbol{U} = \theta((V^\tau)^\perp)$ and let $l = \dim \boldsymbol{U}$.

COROLLARY 2.2. *Any isotropic space is contained in an isotropic space of maximal dimension ν. In addition $\nu = (m - l)/2$.*

PROOF. Given any isotropic space U, we can find a subspace U' of a maximal isotropic space M such that $U\cap(V^\tau)^\perp = U'\cap(V^\tau)^\perp$. Hence there is a σ in $U(f)$ such that $\sigma(U)=U'$ and U is thus contained in the maximal isotropic space $\sigma^{-1}(M)$.

It can be shown that the direct sum of a maximal isotropic space in a complement of $V^\tau\cap(V^\tau)^\perp$ in V^τ and $V^\tau\cap(V^\tau)^\perp$ is a maximal isotropic subspace of V. Hence,

$$\begin{aligned}\nu &= \dim V^\tau \cap (V^\tau)^\perp + \frac{\dim V^\tau - \dim V^\tau \cap (V^\tau)^\perp}{2}\\ &= \frac{\dim V^\tau + \dim V^\tau \cap (V^\tau)^\perp}{2}\\ &= \frac{m - (\dim (V^\tau)^\perp - \dim V^\tau \cap (V^\tau)^\perp)}{2}\\ &= \frac{m-l}{2}.\end{aligned}$$

On p. 51 of [3], Wall has shown that $\langle f(x,x), f(y,y)\rangle = f(x,y)$ for x and y in $(V^\tau)^\perp$ uniquely defines a function $\gamma=\langle\lambda,\mu\rangle$ of the variables λ, μ in $\boldsymbol{U}$. Lemma 3.4.2 of [3] states that two nondegenerate forms are equivalent if, and only if, they have the same $\boldsymbol{W}$, $\boldsymbol{U}$, and γ. If f is a degenerate form let $d(f)=\dim V\cap V^\perp$. Then it follows that two (possibly degenerate) forms are equivalent if, and only if, they have the same $\boldsymbol{W}$, $\boldsymbol{U}$, γ, and d.

If f_1 and f_2 are two forms on V, let V_1^τ denote the V^τ for f_1 and V_2^τ denote the V^τ for f_2.

THEOREM 3. *Two (possibly degenerate) forms f_1 and f_2 are equivalent if, and only if,*
(1) $\{f_1(x,x)\} = \{f_2(x,x)\}$ *and*
(2) $(V_1^\tau)^\perp$ *is isometric to* $(V_2^\tau)^\perp$.

PROOF. By the above discussion on $\boldsymbol{W}$, $\boldsymbol{U}$, γ, and d, it is enough to show that if two forms f_1 and f_2 satisfy conditions (1) and (2) they have the same $\boldsymbol{W}$, $\boldsymbol{U}$, γ, and d.

Clearly if f_1 and f_2 satisfy condition (1) they have the same $\boldsymbol{W}$.

Also if f_1 and f_2 satisfy condition (2) it is not hard to see that they must have the same $\boldsymbol{U}$ and γ.

To see that f_1 and f_2 have the same d note that

$$\begin{aligned}d(f_i) &= \dim V_i^\tau + \dim (V_i^\tau)^\perp - \dim V\\ &= m - \dim \boldsymbol{W} + \dim (V_i^\tau)^\perp - m \qquad i = 1, 2\\ &= \dim (V_i^\tau)^\perp - \dim \boldsymbol{W}. \quad \text{Q.E.D.}\end{aligned}$$

In [2] it was shown that any two nondegenerate forms f_1 and f_2 are equivalent under the condition that $f_1(x, x)$ and $f_2(x, x)$ both take their values in a perfect subfield of D. The next corollary shows that this is true for any two nondegenerate forms whose $\boldsymbol{W}$'s are the same one-dimensional subspace of D.

COROLLARY 3.1. *If f_1 and f_2 are nondegenerate forms, each with a one-dimensional $\boldsymbol{W}$, then f_1 is equivalent to f_2 if, and only if, condition* (1) *holds.*

PROOF. Since $\dim(V_1^\tau)^\perp=\dim \boldsymbol{W}=\dim(V_2^\tau)^\perp$, $\dim(V_1^\tau)^\perp=\dim(V_2^\tau)^\perp=1$. Hence $(V_1^\tau)^\perp$ is isometric to $(V_2^\tau)^\perp$ if, and only if, either both are isotropic or both are not isotropic. It is known (p. 50 of [3]) that $\dim \theta((V_1^\tau)^\perp)\equiv m(2)$ and $\dim \theta((V_2^\tau)^\perp)\equiv m(2)$ so that $(V_1^\tau)^\perp$ and $(V_2^\tau)^\perp$ will be isotropic when m is even, not isotropic when m is odd.

REMARK. If f_1 and f_2 are (possibly degenerate) forms, each with a one-dimensional $\boldsymbol{W}$, then f_1 is equivalent to f_2 if, and only if condition (1) holds and $d(f_1)=d(f_2)$.

THEOREM 4. *Let W_1, W_2 be subspaces of V. Then W_1 is equivalent to W_2 if, and only if:*

(1) $\dim W_1=\dim W_2$, *and*

(2) $W_1\cap(V^\tau)^\perp=W_2\cap(V^\tau)^\perp$, *and*

(3) $W_1^\perp\cap(V^\tau)^\perp=W_2^\perp\cap(V^\tau)^\perp$, *and*

(4) $(W_1\cap V^\tau)^\perp\cap W_1$ *is isometric to* $(W_2\cap V^\tau)^\perp\cap W_2$.

PROOF. Conditions (2) and (3) are necessary by Lemma 1.2.2, Corollary [3]. Conditions (1) and (4) are obviously necessary.

To prove the sufficiency we will first show that conditions (1), (3), and (4) imply that W_1 is isometric to W_2. Then the theorem follows from condition (2) and Theorem 2. Theorem 3 shows that W_1 and W_2 are isometric. This follows since condition (4) gives us condition (2) in Theorem 3 immediately. Condition (3) is equivalent to $W_1+V^\tau=W_2+V^\tau$ which implies condition (1) in Theorem 3.

The next corollary shows that these conditions are somewhat simpler for the situation where $\dim \boldsymbol{W}=1$ and is a generalization of Corollary 3.2 in [2].

COROLLARY 4.1. *If* $\dim \boldsymbol{W}=1$, *then two subspaces W_1 and W_2 of V are equivalent if, and only if:*

(1) $\dim W_1=\dim W_2$, *and*

(2) $\dim W_1\cap(V^\tau)^\perp=\dim W_2\cap(V^\tau)^\perp$, *and*

(3) $\dim W_1\cap V^\tau=\dim W_2\cap V^\tau$, *and*

(4) $\dim W_1\cap W_1^\perp=\dim W_2\cap W_2^\perp$.

PROOF. These conditions are necessary by Theorem 4. The sufficiency is proved in a fashion similar to the proof of Theorem 4.

By (1) and (3) $\dim \theta(W_1) = \dim \theta(W_2)$. Since $\dim \boldsymbol{W} = 1$, either $\dim \theta(W_1)$ and $\dim \theta(W_2)$ are both 1 or both 0. Since condition (3) implies $W_1 + V^\tau = W_2 + V^\tau$, if both dimensions are 1, W_1 and W_2 are isometric by the Remark to Corollary 3.1. In case both dimensions are 0, conditions (1) and (4) are the known conditions for two symplectic spaces to be isometric.

Noting that condition (2) implies $W_1 \cap (V^\tau)^\perp = W_2 \cap (V^\tau)^\perp$, Theorem 2 tells us that W_1 and W_2 are equivalent.

REMARK. If $(V^\tau)^\perp = 0$, Theorems 3 and 4 are known [1] theorems for nondegenerate, alternating forms on spaces over fields of any characteristic.

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