ON A PROBLEM CONNECTED WITH THE VANDERMONDE DETERMINANT

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Let $m$ be a positive integer. We denote by $a = (a_0, a_1, \ldots, a_m)$, $x = (x_0, \ldots, x_{m-1})$ vectors with real components. Define

$$U = \{ x \mid \text{all } |x_i| \leq 1, x_i \neq x_j, \ i \neq j \},$$

$$P^*(x, a) = \begin{vmatrix}
1 & 1 & \cdots & 1 & a_0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
x_0 & x_1 & \cdots & x_{m-1} & a_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
x_0 & x_1 & \cdots & x_{m-1} & a_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
x_0 & x_1 & \cdots & x_{m-1} & a_m
\end{vmatrix},$$

$$A^* = \{ a \mid P^*(x, a) \neq 0 \text{ for every } x \in U \}.$$

This paper is devoted to the problem of characterizing $A^*$, a problem which is of interest in the theory of the design of statistical experiments (see [1]) and is perhaps of interest per se.

When $m = 1$, it is trivial that $A^* = \{ (a_0, a_1) \mid |a_0| < |a_1| \}$. We hereafter assume $m > 1$. Define

$$A(c) = A^* \cap \{ a \mid a_m = c \}.$$

We shall prove the following:

(I) $A(1)$ is a set $A$ described precisely below.

(II) For $c \neq 0$, we have $(a_0, a_1, \ldots, a_{m-1}, c) \subseteq A(c)$ if and only if $(a_0/c, \ldots, a_{m-1}/c, 1) \subseteq A$.

(III) $A(0)$ is empty.

Of these, (II) is obvious, and (III), which we shall prove in the paragraph after next, is only slightly less so. The problem is to describe $A$.

Let $S_0(x) = 1$ and

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(2) \[ S_j(x) = (-1)^j \sum x_{i_1}x_{i_2} \cdots x_{i_j}, \]
where the summation is with respect to \( i_1, \ldots, i_j, \) \( 0 \leq i_1 < i_2 < \cdots < i_j < m. \) Let

(3) \[ P(x, a) = \sum_{j=0}^{m} a_{m-j} S_j(x). \]

Expanding the determinant \( P* \) of (1) in minors of the last column shows that

\[ P*(x, a) = P(x, a) K(x), \]

where the polynomial \( K \) is the Vandermonde determinant which is the minor of \( a_m \) in the determinant \( P* \) of (1); thus, \( K \) is never zero on \( U. \) Hence in the definition of \( A* \) we may replace \( P*(x, a) \) by \( P(x, a). \)

We shall now prove that \( A(0) \) is empty. Let \( a \) be any \((m+1)\)-vector with \( a_m = 0. \) If all \( a_j = 0, \) clearly \( a \not\in A*. \) Suppose then that \( j_0 \)

is the smallest integer for which \( a_{m-j_0} \neq 0, \) \( 0 < j_0 \leq m. \) We shall find two points \( x' \) and \( x'' \) in the convex subset \( T \) of \( U \) where \(-1 \leq x_0 < x_1 < \cdots < x_{m-1} \leq 1 \) such that

(4) \[ (-1)^{j_0} a_{j_0} P(x', a) < 0 < (-1)^{j_0} a_{j_0} P(x'', a), \]

from which it follows that \( P(x, a) = 0 \) for some \( x \) in \( T; \) it follows that \( A(0) \) is empty. Let \( x'' = (\epsilon, 2\epsilon, \cdots, m\epsilon) \) with \( 0 < \epsilon < 1. \) Each term in the sum in (2) is then positive, and thus \( (-1)^{j_0} a_{j_0} P(x'') > \epsilon k, \) while \( S_j(x'') = o(\epsilon k) \) as \( \epsilon \to 0 \) if \( j > j_0. \) Hence, for \( \epsilon \) sufficiently small, the second half of (4) follows from (3). The first half of (4) follows similarly if \( j_0 \) is odd upon taking \( x' = (-m\epsilon, -(m-1)\epsilon, \cdots, -\epsilon), \) and if \( j_0 \) is even upon taking \( x' = (-m-1)\epsilon, -(m-2)\epsilon, \cdots, -\epsilon, 1). \)

Let

(5) \[ Q_h(x) = \sum (1 - x_{i_1})(1 - x_{i_2}) \cdots (1 - x_{i_h})(1 + x_{i_{h+1}}) \cdots (1 + x_{i_m}) / \binom{m}{h}, \]

where the summation is over the \( \binom{k}{h} \) choices of the disjoint sets \( (i_1, i_2, \cdots, i_h) \) and \( (i_{h+1}, \cdots, i_m) \) of distinct integers between 0 and \( m-1, \) inclusive. Define the points

(6) \[ a^{(h)} = (a_0^{(h)}, \cdots, a_m^{(h)}), \]

by
We note that $a_m^{(h)} = 1$ for all $h$, and that

$$a^{(0)} = ((-1)^m, (-1)^{m-1}, \ldots, (-1)^0),$$

$$a^{(m)} = (1, 1, \ldots, 1).$$

We will now prove:

**Theorem.** For $m > 1$, $A$ is the closed $m$-simplex with extreme points $a^{(0)}, \ldots, a^{(m)}$, minus the (closed) edge which connects $a^{(0)}$ and $a^{(m)}$.

Let $\bar{U}$ be the closure of $U$. If $x$ is a vertex of $\bar{U}$ with $r$ coordinates $-1$ and $(m-r)$ coordinates $+1$, then

$$Q_h(x) = \begin{cases} 2^m \binom{m}{h} & \text{if } h = r, \\ 0 & \text{if } h \neq r. \end{cases}$$

Hence the $Q_h$ are linearly independent functions. Moreover, each $Q_h(x) \geq 0$ on $U$. If $h \neq 0$ or $m$, $Q_h(x) > 0$ on $U$, since at most one $x_i$ can take the value $+1$ and at most one $x_i$ can take the value $-1$ on $U$.

According to (3), $P(x, a)$ is linear in $a$. According to (7), $P(x, a^{(h)}) = Q_h(x)$. Combining these facts, if $\alpha_0, \ldots, \alpha_m$ are real numbers,

$$P \left( x, \sum_{h=0}^{m} \alpha_h a^{(h)} \right) = \sum_{h=0}^{m} \alpha_h Q_h(x).$$

(8)

Since the $Q_h$ are linearly independent, it follows from (7) that the vectors $a^{(i)}$, $0 \leq i \leq m$, are linearly independent. Hence, $a^{(0)}, \ldots, a^{(m)}$ lie in the hyperplane $a_m = 1$, but in no other hyperplane. Hence, any point $a$ in the hyperplane $a_m = 1$ can be written in a unique way as

$$a = \sum_{h=0}^{m} \alpha_h a^{(h)},$$

where $\sum \alpha_h = 1$. Suppose now that $\alpha_h \geq 0$ for all $h$, and $\alpha_h > 0$ for some $h \neq 0$ or $m$. Then, from (8) and the paragraph below the statement of the theorem, we have $P(x, a) > 0$ for all $x$ in $U$. From this it follows that $A$ is at least as large as stated in the theorem. It remains to prove that it is no larger.

If $\alpha_0 + \alpha_m = 1$ and all other $\alpha_h = 0$, we have $P(x, a) = 0$ for any $x$ in $U$ such that $x_0 = 1$, $x_1 = -1$. Hence the closed segment $[a^{(0)}, a^{(m)}]$ is not in $A$. 

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Suppose now that $\alpha_{i_0} < 0$. If $0$ is the origin then of course $P(0, a) = 1$. It follows from the paragraph below the statement of the theorem that there is an $x^*$ in $U$ such that (i) the half-open segment $(0, x^*]$ is in $U$, (ii) $Q_{i_0}(x^*) > \frac{1}{2}$, and (iii) for $j \neq i_0$, $Q_j(x^*) < \frac{1}{2}$. From (8) we have $P(x^*, a) < 0$. Hence, for some point $y \in (0, x^*)$ (so that $y \in U$) we must have $P(y, a) = 0$. Hence $a$ cannot be in $A$. The proof of the theorem is complete.

We remark that the points $a$ of the form $(b^m, b^{m-1}, \ldots, b^1, 1)$ with $|b| < 1$ are easily verified to be in $A$; this is of interest in applications.

Reference


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