ON A PROBLEM CONNECTED WITH THE VANDERMONDE DETERMINANT

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Let m be a positive integer. We denote by $a=(a_0, a_1, \cdots, a_m)$, $x=(x_0, \cdots, x_{m-1})$ vectors with real components. Define

$$U = \{x \mid \text{all } |x_i| \leqq 1,\ x_i \neq x_j,\ i \neq j\},$$

$$(1) \qquad P^*(x, a) = \begin{vmatrix} 1 & 1 & \cdot & 1 & a_0 \\ x_0 & x_1 & \cdot & x_{m-1} & a_1 \\ x_0^2 & x_1^2 & \cdot & x_{m-1}^2 & a_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_0^m & x_1^m & \cdot & x_{m-1}^m & a_m \end{vmatrix},$$

$$A^* = \{a \mid P^*(x, a) \neq 0 \text{ for every } x \in U\}.$$

This paper is devoted to the problem of characterizing A^*, a problem which is of interest in the theory of the design of statistical experiments (see [1]) and is perhaps of interest per se.

When $m=1$, it is trivial that $A^*=\{(a_0, a_1) \mid |a_0| < |a_1|\}$. We hereafter assume $m>1$. Define

$$A(c) = A^* \cap \{a \mid a_m = c\}.$$

We shall prove the following:

(I) $A(1)$ is a set A described precisely below.

(II) For $c \neq 0$, we have $(a_0, a_1, \cdots, a_{m-1}, c) \in A(c)$ if and only if $(a_0/c, \cdots, a_{m-1}/c, 1) \in A$.

(III) $A(0)$ is empty.

Of these, (II) is obvious, and (III), which we shall prove in the paragraph after next, is only slightly less so. The problem is to describe A.

Let $S_0(x) \equiv 1$ and

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$$S_j(x) \equiv (-1)^j \sum x_{i_1} x_{i_2} \cdots x_{i_j}, \tag{2}$$

where the summation is with respect to $i_1, \cdots, i_j$, $0 \leqq i_1 < i_2 < \cdots < i_j < m$. Let

$$P(x, a) \equiv \sum_{j=0}^{m} a_{m-j} S_j(x). \tag{3}$$

Expanding the determinant P^* of (1) in minors of the last column shows that

$$P^*(x, a) = P(x, a) K(x),$$

where the polynomial K is the Vandermonde determinant which is the minor of a_m in the determinant P^* of (1); thus, K is never zero on U. Hence in the definition of A^* we may replace $P^*(x, a)$ by $P(x, a)$.

We shall now prove that $A(0)$ is empty. Let a be any $(m+1)$-vector with $a_m = 0$. If all $a_j = 0$, clearly $a \notin A^*$. Suppose then that j_0 is the smallest integer for which $a_{m-j_0} \neq 0$, $0 < j_0 \leqq m$. We shall find two points x' and x'' in the convex subset T of U where $-1 \leqq x_0 < x_1 < \cdots < x_{m-1} \leqq 1$ such that

$$(-1)^{j_0} a_{j_0} P(x', a) < 0 < (-1)^{j_0} a_{j_0} P(x'', a), \tag{4}$$

from which it follows that $P(x, a) = 0$ for some x in T; it follows that $A(0)$ is empty. Let $x'' = (\epsilon, 2\epsilon, \cdots, m\epsilon)$ with $0 < \epsilon < 1$. Each term in the sum in (2) is then positive, and thus $(-1)^{j_0} S_{j_0}(x'') > \epsilon^{j_0}$, while $S_j(x'') = o(\epsilon^{j_0})$ as $\epsilon \to 0$ if $j > j_0$. Hence, for ϵ sufficiently small, the second half of (4) follows from (3). The first half of (4) follows similarly if j_0 is odd upon taking $x' = (-m\epsilon, -(m-1)\epsilon, \cdots, -\epsilon)$, and if j_0 is even upon taking $x' = (-(m-1)\epsilon, -(m-2)\epsilon, \cdots, -\epsilon, 1)$.

Let

$$Q_h(x) \equiv \sum (1 - x_{i_1})(1 - x_{i_2}) \cdots (1 - x_{i_h})(1 + x_{i_{h+1}}) \cdots \cdot (1 + x_{i_m}) \Big/ \binom{m}{h}, \qquad 0 \leqq h \leqq m, \tag{5}$$

where the summation is over the $\binom{m}{h}$ choices of the disjoint sets $(i_1, i_2, \cdots, i_h)$ and $(i_{h+1}, \cdots, i_m)$ of distinct integers between 0 and $m-1$, inclusive. Define the points

$$a^{(h)} = (a_0^{(h)}, \cdots, a_m^{(h)}), \qquad 0 \leqq h \leqq m, \tag{6}$$

by

(7) $$Q_h(x) = \sum_{j=0}^{m} a_{m-j}^{(h)} S_j(x).$$

We note that $a_m^{(h)} = 1$ for all h, and that

$$a^{(0)} = ((-1)^m, (-1)^{m-1}, \cdots, (-1)^0),$$
$$a^{(m)} = (1, 1, \cdots, 1).$$

We will now prove:

THEOREM. *For $m > 1$, A is the closed m-simplex with extreme points $a^{(0)}, \cdots, a^{(m)}$, minus the (closed) edge which connects $a^{(0)}$ and $a^{(m)}$.*

Let $\bar{U}$ be the closure of U. If x is a vertex of $\bar{U}$ with r coordinates -1 and $(m-r)$ coordinates $+1$, then

$$\begin{aligned} Q_h(x) &= 2^m \Big/ \binom{m}{h} && \text{if } h = r, \\ &= 0 && \text{if } h \neq r. \end{aligned}$$

Hence the Q_h are linearly independent functions. Moreover, each $Q_h(x) \geqq 0$ on U. If $h \neq 0$ or m, $Q_h(x) > 0$ on U, since at most one x_i can take the value $+1$ and at most one x_i can take the value -1 on U.

According to (3), $P(x, a)$ is linear in a. According to (7), $P(x, a^{(h)}) = Q_h(x)$. Combining these facts, if $\alpha_0, \cdots, \alpha_m$ are real numbers,

(8) $$P\left(x, \sum_0^m \alpha_h a^{(h)}\right) = \sum_0^m \alpha_h Q_h(x).$$

Since the Q_h are linearly independent, it follows from (7) that the vectors $a^{(i)}$, $0 \leqq i \leqq m$, are linearly independent. Hence, $a^{(0)}, \cdots, a^{(m)}$ lie in the hyperplane $a_m = 1$, but in no other hyperplane. Hence, any point a in the hyperplane $a_m = 1$ can be written in a unique way as

$$a = \sum_0^m \alpha_h a^{(h)},$$

where $\sum \alpha_h = 1$. Suppose now that $\alpha_h \geqq 0$ for all h, and $\alpha_h > 0$ for some $h \neq 0$ or m. Then, from (8) and the paragraph below the statement of the theorem, we have $P(x, a) > 0$ for all x in U. From this it follows that A is at least as large as stated in the theorem. It remains to prove that it is no larger.

If $\alpha_0 + \alpha_m = 1$ and all other $\alpha_h = 0$, we have $P(x, a) = 0$ for any x in U such that $x_0 = 1$, $x_1 = -1$. Hence the closed segment $[a^{(0)}, a^{(m)}]$ is not in A.

Suppose now that $\alpha_{i_0} < 0$. If 0 is the origin then of course $P(0, a) = 1$. It follows from the paragraph below the statement of the theorem that there is an x^* in U such that (i) the half-open segment $(0, x^*]$ is in U, (ii) $Q_{i_0}(x^*) > \frac{1}{2}$, and (iii) for $j \neq i_0$, $Q_j(x^*) < |\alpha_{i_0}| / 2 \sum |\alpha_h|$. From (8) we have $P(x^*, a) < 0$. Hence, for some point $y \in (0, x^*)$ (so that $y \in U$) we must have $P(y, a) = 0$. Hence a cannot be in A. The proof of the theorem is complete.

We remark that the points a of the form $(b^m, b^{m-1}, \cdots, b^1, 1)$ with $|b| < 1$ are easily verified to be in A; this is of interest in applications.

Reference

1. J. Kiefer and J. Wolfowitz, *On a theorem of Hoel and Levine on extrapolation designs*, Ann. Math. Statist. (to appear).

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