If $F$ is a ring, then an obvious way to construct subrings of $(F)_n$, the ring of all $n \times n$ matrices over $F$, is to choose additive subgroups $F_{ij}$ of $F$ such that

$$F_{ij}F_{ik} \subset F_{ik}, \quad i, j, k = 1, \ldots, n,$$

and then form the ring

$$R = \sum_{i,j=1}^n F_{ij}e_{ij}$$

where the $e_{ij}$ are the usual unit matrices. For example, we could select $n$ left ideals $A_1, \ldots, A_n$ of either $F$ or a subring of $F$ and then let $F_{ij} = A_j$, $i, j = 1, \ldots, n$.

If $F$ is a (skew) field and the $F_{ij}$ satisfying (1) are all nonzero, then $R$ defined by (2) is easily shown to be a prime ring. The main result of this paper (1.3) is that if $F$ is a right ring of quotients of $F_{11}$ then $(F)_n$ is a right ring of quotients of $R$ and there exists a subring $K$ of $F$ and a nonzero diagonal matrix $d \in R$ such that $(K)_n$ is a subring of $dRd^{-1}$ and $F$ is a ring of quotients of $K$. This result is used to give new proofs of the Faith-Utumi theorem [2] and of Goldie’s theorem [1].

1. Prime matrix rings. If $A$ is a subset of a ring, then let $A' = \{ x \in A \mid x \neq 0 \}$. If $A$ and $B$ are subsets of a field, then denote by $AB^{-1} = \{ ab^{-1} \mid a \in A, b \in B' \}$. The notation $R \subseteq S$ is used to show that $S$ is a right ring of quotients of $R$; that is, that $R$ is a subring of $S$ and a $R \cap R \neq 0$ for all $a \in S'$. It is readily seen that if $F$ is a field and $K$ is a subring of $F$, then $K \subseteq F$ iff $KK^{-1} = F$.

1.1. Lemma. Let $F$ be a field and $A$ be a subring of $F$ for which $AA^{-1} = F$. If $B$ and $C$ are nonzero right $A$-modules contained in $F$, then $B \cap C \neq 0$ and $BC^{-1} = F$.

Proof. For any $f \in F'$, $b \in B'$, and $c \in C'$, there exist $a_i \in A$ such that $b^{-1}fc = a_0a_1^{-1}$. Hence, $f = (ba_1)(ca_2)^{-1} \in BC^{-1}$. We conclude that $BC^{-1} = F$. If $f = 1$, then $ba_1 = ca_2$ and evidently $B \cap C \neq 0$.

1.2. Theorem. Let $F$ be a field, $\{ F_{ij} \mid i, j = 1, \ldots, n \}$ be a set of
nonzero additive subgroups of $F$ satisfying (1), and $R$ be the prime ring defined by (2). Then $R \subseteq (F)$ if and only if $F_1 \subseteq F$.

Proof. If $R \subseteq (F)$, then for every $d \in F'$ there exists $a \in R$ such that $(de_1)a \in R^1$. If $a = \sum a_{ij}e_{ij}$, then $da_{ij} \in F_{ij}$ and $d \in F_{ij}F_{ij}^{-1}$ for some $j$. Since $f_{ij}^{-1} = (fh)(gh)^{-1}$ for all $f, g \in F_{ij}$ and $h \in F_{ji}$, evidently $F_{ij}F_{ij}^{-1} \subseteq F_1F_{ij}^{-1} \subseteq F_{ij}$. Hence, $F \subseteq F_1F_{ij}^{-1}$ and $F_1 \subseteq F$.

Conversely, if $F_1 \subseteq F$ then $F_1F_{ij}^{-1} = F$ for all $i$ and $j$ by (1.1). Actually, $fg^{-1} = (fh)(gh)^{-1}$ for all $f \in F_{ij}$, $g \in F_{ji}$, $h \in F_{ik}$ so that $F_{ik}F_{ik}^{-1} = F$ for all $i, j, k$. Since each $F_{ik}$ is a right $F_{kk}$-module, the $F_{kk}$-module $F_k = \cap_{i=1}^n F_{ik}$ is nonzero and $F_kF_{kk}^{-1} = F$ by (1.1). Clearly each $F_k$ is a subring of $F$. Thus, $R$ contains a subring $S$ of the form

$S = \sum_{i,j=1}^n F_{ij}e_{ij}$

where the $F_{ij}$ are subrings of $F$ satisfying

(4) $F_iF_j \subseteq F_{ij}, \quad F_i \subseteq F, \quad i, j = 1, \ldots, n.$

To complete the proof of 1.2, we need only prove that $S \subseteq (F)$. Let $a = \sum a_{ij}e_{ij} \in (F)$, with $a_{ij} \neq 0$. Then there exist $b \in F_{ij}$ such that $a_{ij}b \in F_i$ for each $i$. Since $b_1 \cap \cdots \cap b_n F_i \neq 0$ by (1.1), there exists some $b \in F_{ij}$ such that $a_{ij}b \in F_i$ for each $i$. Hence, $a(b_{ei}) = \sum a_{ij}b_{ei} \in S$. Therefore, $S \subseteq (F)$ and 1.2 is proved.

It is easy to give an example showing that $F_1 \cap \cdots \cap F_n$ might be zero in ring $S$ of (3). Thus, if $D$ is a right Ore domain having a quotient field of $F$ (i.e., $DD^{-1} = F$) but $D$ is not a left Ore domain (i.e., $D^{-1}D \neq F$), then there exist nonzero left ideals $F_i$ of $D$ such that $F_1 \cap \cdots \cap F_n = 0$. Still, $S \subseteq (F)$ if $S$ is defined by (3). Although the intersection of the $F_k$ might be zero, the intersection of the corresponding subrings of $F$ in some isomorphic image of $S$ in $(F)$ is nonzero as we shall now show.

Let $S$ be a subring of $(F)$ defined by (3) and (4) above, and let $g_i \in F_i$, $i = 1, \ldots, n$. If $f_k = g_1g_2 \cdots g_k$, $k = 1, \ldots, n$, then clearly each $f_k \in F_k$ and $d = \sum f_{ei}e_{ii}$ has inverse $d^{-1} = \sum f_{ij}^{-1}e_{ii}$ in $(F)$. Let

$T = dSd^{-1} = \sum_{i,j=1}^n (f_if_{ij}^{-1})e_{ij}$

an isomorphic image of $S$. Evidently $T \subseteq (F)$. Since $f_{a}a_{ij}^{-1} = f_{a}a_{ij}^{-1} \cdots g_s a g_s \cdots g_{k}^{-1} \in F_{ij}$ for all $a \in F_i$, clearly

$\bigcap_{i,j=1}^n F_{ij}^{-1} = F_1F_{ij}^{-1}.$
If we let $K = f_n F_i f_i^{-1}$, then $K$ is a subring of $F$ for which $KK^{-1} = F$ by 1.1. Evidently $(K)_n \subseteq T$ and also $(K)_n \subseteq (F)_n$ by 1.2. We have proved the following result.

1.3. Theorem. Let $F$ be a field, $F_i$ be nonzero additive subgroups of $F$ satisfying (1), and $R$ be the prime ring defined by (2). If $F_i \subseteq F$, then there exists a subring $K$ of $F$ and a nonsingular diagonal matrix $d \in R$ such that $KK^{-1} = F$ and $(K)_n \subseteq dRd^{-1} \subseteq (F)_n$.

2. The annihilator ideal lattice. In order to apply the theorems of §1 to prime rings in general, we need the following lattice-theoretic results. Since we wish to use these results in another context [7], we shall state them in as general terms as possible.

Let $P$ be a ring, $L_r$ be the lattice of right ideals of $P$, and $R^s_r$ be the right singular ideal of $P$. Thus, $b \in R^s_r$ iff $b^r = \{ x \in R \mid bx = 0 \}$ is a large right ideal; i.e., $b^r \cap A \neq 0$ for all nonzero $A \in L_r$. If $R^s_r = 0$, then we denote the lattice of closed right ideals of $R$ by $L^s_r$. Thus, $A \in L^s_r$ iff $A$ is the only essential extension of $A$ in $L_r$. It is well-known that $L^s_r$ is a complete complemented modular lattice.

If $L$ is a lattice containing $0$ and $I$, then a minimal (maximal) element of $L - \{ 0 \} (L - \{ I \})$ is called an atom (coatom). If $P = 0$ and $L^s_r$ is atomic (i.e., every nonzero element of $L^s_r$ contains an atom), then let us denote by $R^s_t$ the union in $L_r$ of all atoms of $L^s_r$. A ring $R$ is called (right) stable [6] iff $R^s_t = 0$, $L^s_r$ is atomic, and $(R^s_t)^r = 0$. Not only is every prime ring (for which $L^s_r$ is atomic) stable, but so also is every $n \times n$ triangular matrix ring over a right Ore domain.

Another lattice associated with a ring $R$ is the lattice $J_r$ of annihilating right ideals of $R$. If $R^s_t = 0$ then $J_r$ is a subset, although not necessarily a sublattice, of $L^s_r$. However, intersections are set-theoretic in both lattices.

Needless to say, the corresponding left structure of a ring $R$ is indicated by replacing each "r" above by an "l".

The following lemma, due to Koh [3], is basic to the work of this section. Our proof is a paraphrase of Koh’s proof for prime rings.

2.1. Lemma. If $R$ is a stable ring then $R^s_t = 0$.

Proof. If $R^s_t \neq 0$ and $d \in (R^s_t)^r$, then $Ad \neq 0$ for some atom $A \in L^s_r$ and $ad \neq 0$ for some $d \subseteq A^r$. Since $Ra \cap d^r \neq 0$, $xa \neq 0$ and $xad = 0$ for some $x \subseteq R^r$. However, $a^r$ is a coatom of $L^s_r$ by [4, 6.9] and therefore $(xa)^r = a^r$. This contradiction proves the lemma.

The lattices $J_r$ and $J_i$ are dual isomorphic under the correspondence $A \leftrightarrow A^r, A \subseteq L_r$. If $R$ is a stable ring then the lattice $J_i$ is atomic. Actually, let us show that if $A, B \subseteq J_i$ with $A \cap B \neq B$, then there
exists an atom \( C \in J_1 \) such that \( C \subseteq B \) and \( C \cap A = 0 \). By 2.1, \( L^*_1 \supseteq J_1 \) and there exists some nonzero \( D \in L^*_1 \) such that \( D \subseteq B \) and \( D \cap A = 0 \). Since \( R \) is stable, \( ED \neq 0 \) and hence \( E \cap D \neq 0 \) for some atom \( E \in L^*_1 \). If \( d \in (E \cap D)' \), then \( d^* \) is a coatom of \( L^*_1 \) by \([4, 6.9]\). Therefore, \( d^* \) is a coatom of \( J_1 \) and \( C = d^* \) is an atom of \( J_1 \). Clearly \( C \subseteq B \) and \( C \cap A = 0 \).

If \( B \) is any atom of \( J_1 \) then \( B^r \) is a coatom of \( L^*_1 \) by the proof above. Thus, if \( B \) covers 0 in \( J_1 \) then \( 0^r = R \) covers \( B^r \) in \( L^*_1 \). This is a special case of the following result.

2.2. Lemma. Let \( R \) be a stable ring and \( A, B \in J_1 \). Then \( B \) is a cover of \( A \) in \( J_1 \) iff \( A^r \) is a cover of \( B^r \) in \( L^*_1 \).

Proof. If \( A^r \) is a cover of \( B^r \) in \( L^*_1 \), then \( A^r \) is a cover of \( B^r \) in \( J_1 \) and \( B \) is a cover of \( A \) in \( J_1 \). Conversely, if \( B \) is a cover of \( A \) in \( J_1 \) then there exists an atom \( C \in J_1 \) such that \( C \subseteq B \) and \( C \cap A = 0 \). Clearly \( B = A \cup C \) in \( J_1 \). Hence, \( B^r = A^r \cap C^r \). Since \( C^r \) is a coatom of \( L^*_1 \) and \( A^r \subseteq C^r \), evidently \( A^r \cup C^r = R \) in \( L^*_1 \). Therefore, the intervals \([C^r, R] \) and \([A^r \cap C^r, A^r] \) are isomorphic and \( A^r \) covers \( A^r \cap C^r = B^r \).

The main result of this section is as follows.

2.3. Theorem. If \( R \) is a stable ring then the lattice \( J_1 \) is upper semimodular.

Proof. Let \( A, B \in J_1 \) be covers of \( A \cap B \). Then \((A \cap B)^r \) covers \( A^r \) and \( B^r \) in \( L^*_1 \) by 2.2. Hence, \((A \cap B)^r = A^r \cup B^r \) in \( L^*_1 \). By the modularity of \( L^*_1 \), \( A^r \) and \( B^r \) cover \( A^r \cap B^r \). Therefore, \( A \cup B \) \((= (A \cap B^r)^r) \) covers \( A \) and \( B \) in \( J_1 \) by 2.2. This proves 2.3.

If the lattice \( L^*_1 \) has a finite dimension \( n \), then we call \( n \) the (right) dimension of \( R \) and write \( \dim R = n \).

2.4. Corollary. If \( R \) is a stable ring such that \( \dim R = n \), then \( J_1 \) is a complemented lattice of dimension \( n \).

Proof. Every maximal chain in \( J_1 \) has length by 2.2, and therefore \( \dim J_1 = n \). To show that \( J_1 \) is complemented, let \( A, B \in J_1 \) with \( A \cap B = 0 \) and \( A \cup B \neq R \). Then there exists an atom \( C \in J_1 \) such that \( C \cap (A \cup B) = 0 \). We claim that \( A \cap (B \cup C) = 0 \) in \( J_1 \). If this is so, then by induction there exists some \( D \in J_1 \) such that \( A \cap D = 0 \) and \( A \cup D \neq R \). Hence, \( J_1 \) is complemented.

If \( A \cap (B \cup C) \neq 0 \), then there exists an atom \( E \in J_1 \) such that \( E \subset A \cap (B \cup C) \). Then \( E \cap B = 0 \), \( E^r \supseteq A^r \), and \( E^r \supseteq B^r \cap C^r \). Clearly \( C^r \cup (A^r \cap B^r) = R \) in \( L^*_1 \). Hence, \( B^r = B^r \cap (C^r \cup (A^r \cap B^r)) \) = \( (B^r \cap C^r) \cup (A^r \cap B^r) \) and \( E^r \supseteq B^r \), contrary to the fact that \( E \cap B = 0 \). Hence, \( A \cap (B \cup C) = 0 \).
The lattice $J_i$ of a stable ring $R$ is not necessarily modular, as the following example shows.

2.5. Example. Let $D$ be a right Ore domain which is not a left Ore domain, $F$ be the right field of quotients of $D$, and $R = (D)_{\mathbb{F}}$. Clearly $R$ is a prime ring of dimension 3. Select $g, h \in D'$ such that $Dg \cap Dh = 0$, and let $u = ge_{11} + e_{21}, v = e_{21} + he_{11}$ in $R$. Then $u' = Re_{21} + R(e_{11} - ge_{21})$ and $v' = Re_{11} + R(he_{12} - e_{13})$. Since $re_{11}$ and $re_{21}$ are atoms of $J_i$, evidently $u'$ and $v'$ are coatoms of $J_i$. However, $u' \cup v' = R$ and $U' \cap v' = 0$, and therefore $J_i$ is not modular (since it is not lower semi-modular).

3. Goldie prime rings. A prime ring $R$ such that $R^\times = 0$ and dim $R = n > 1$ is called a Goldie prime ring. Such rings were studied by Goldie in [1]. By 2.4, $J_i$ is a complemented, upper semi-modular lattice for such a ring.

Let $P$ be a Goldie prime ring and $n = \dim R$. By 2.4, there exists an independent set $\{B_1, \ldots, B_n\}$ of atoms of $J_i$ (i.e., $(B_i \cup \cdots \cup B_j) \cap B_{i+1} = 0, i = 1, \ldots, n - 1$). Hence, $\{B_1, \ldots, B_n\}$ is an independent set of coatoms of $L^*_i$ (i.e., $(B_i \cap \cdots \cap B_j) \cup B_{i+1} = R, i = 1, \ldots, n - 1$). If we let

$$A_j = \bigcap_{i=1, i \neq j}^n B_i, \quad j = 1, \ldots, n,$$

then we may show lattice-theoretically that $\{A_1, \ldots, A_n\}$ is an independent set of atoms of $L^*_i$. What is more important, the $A_i$ are in $J_i$. Clearly

$$B_i^* = \bigcup_{j=1, j \neq i}^n A_i, \quad i = 1, \ldots, n.$$ 

A Goldie prime ring $R$ of dimension $n$ has a full ring $Q$ of linear transformations of an $n$ dimensional vector space over a field as a ring of quotients. This is a weaker result than Goldie’s theorem [1, Theorem 11]. It is well-known that the lattices $L^*_i(Q)$ and $L^*_i(R)$ are isomorphic under the correspondence $B \to B \cap R, B \in L^*_i(Q)$. (See [5] for proofs.)

Corresponding to the independent set $\{A_1, \ldots, A_n\}$ of atoms of $L^*_i(R)$ defined above is an independent set $\{C_1, \ldots, C_n\}$ of atoms of $L^*_i(Q)$. By [8, Proposition 5, p. 52], there exists a set $\{e_{ij}\}$ of $n^2$ matrix units in $Q$ such that $C_i = e_{ii}Q, i = 1, \ldots, n$. Hence, $A_i = (e_{ii}Q) \cap R$ and $B_i^* = (\sum_{j \neq i} e_{ij}Q) \cap R, i = 1, \ldots, n$. Relative to the chosen set of matrix units of $Q$, we can find a field $F$ commuting with the $e_{ij}$ such that [5, Proposition 6, p. 52]
\[ Q = \sum_{i,j=1}^{n} F_{ij} \cong (F)_n. \]

Since \( B_i' \) (in \( R \)) = \( B_i \) (in \( Q \)) \cap R and \( B_i' \) (in \( Q \)) \( \subseteq \) \( L_i^*(Q) \), evidently \( B_i' \) (in \( Q \)) \( \subseteq \) \( \mathbb{Q} e_{ii} \cap R \) for each \( i \). Actually, \( B_i = \mathbb{Q} e_{ii} \cap R \) for each \( i \) since \( [\mathbb{Q} e_{ii} \cap R] B_i' = 0 \). Since \( A_i B_j \neq 0 \) for all \( i \) and \( j \), we see that

\[ A_i \cap B_j = F_{ij} e_{ij}, \quad i, j = 1, \ldots, n \]

for some nonzero additive subgroups \( F_{ij} \) of \( F \) satisfying (1). Hence,

\[ S = \sum_{i,j=1}^{n} F_{ij} e_{ij} \]

is a prime subring of \( R \).

Each nonzero left ideal of \( R \) has \( R \) as a right ring of quotients. In particular, \( B_i \leq R \) and \( L_i^*(B_i) \cong L_i^*(R) \). Therefore, \( \{ F_{11} e_{11}, \ldots, F_{n1} e_{n1} \} \) is an atomic basis of \( L_i^*(B_i) \) and \( F_{11} e_{11} + \cdots + F_{n1} e_{n1} \leq B_i \leq R \). Consequently, \( S \leq R \leq (F)_n \). Now we can apply 1.3 to obtain the following result.

**3.1. Faith-Utumi Theorem.** Every Goldie prime ring \( R \) of dimension \( n \) has associated with it a field \( F \) and a subring \( K \) of \( F \) such that \( K \leq F \) and \( (K)_n \leq R \leq (F)_n \).

An immediate corollary of 3.1 is Goldie's theorem, which states that \( (F)_n = \{ a b^{-1} | a, b \in R, b \ regular \} \). In fact, the following stronger result (due to Faith) holds.

**3.2. Theorem.** If \( R \) is a Goldie prime ring of dimension \( n \) and \( F \) is its associated field, then there exists a subring \( K \) of \( F \) such that

\[ (F)_n = \{ a k^{-1} | a \in R, k \in K' \}. \]

**Proof.** If \( c \in (F)_n \), say \( c = \sum a_{ij} b_{ij}^{-1} e_{ij} \) where \( a_{ij}, b_{ij} \in K \), then \( c k = a \in (K)_n \) for any nonzero \( k \in \cap b_{ij} K \) and \( c = ak^{-1} \) as desired.

**Bibliography**

POWERS IN EIGHTH-GROUPS

SEYMOUR LIPSCHUTZ

1. Introduction. The purpose of this paper is to give an algorithm which decides whether or not an element in an eighth-group is a power. A group $G$ is an eighth-group if it is finitely presented in the form

$$G = \langle a_1, \ldots, a_n; R_1(a_\lambda) = 1, \ldots, R_m(a_\lambda) = 1 \rangle,$$

where (i) each defining relator is cyclically reduced and (ii) if $B_i$ and $B_j$ are cyclic transforms of $R_i$ and $R_j$, then less than one-eighth of the length of the shorter one cancels in the product $B_i^{m_1}B_j^{m_2}$, unless the product is unity. The notation in this paper is the same as that in [3]. Note that Lemma 3 and Lemma 4 in [3] hold for eighth-groups.

Reinhart [4] gives an algorithm to decide, among other things, whether or not elements in certain Fuchsian groups are powers. Note that the Fuchsian group $F(p; n_1, \ldots, n_d; m)$, see Greenberg [1], is an eighth-group if

$$4p + d + m, n_1, \ldots, n_d > 8.$$

Hence our algorithm holds for a somewhat wider class of groups and, furthermore, is purely algebraic.

Remark. Given any word $V$ in a finitely presented group, it is possible to find a cyclically fully reduced word $V^*$ conjugate to $V$ by writing the word $V$ in a circle and then reducing. Such a word $V^*$ will be called a reduced cyclic transform of $V$.

2. The algorithm. First we prove a lemma about eighth-groups $G$. Here $r$ denotes the length of the largest defining relator in $G$.

Received by the editors April 3, 1964.