MAXIMAL QUOTIENT RINGS[1]

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Let R be an associative ring in which an identity element is not assumed. A right quotient ring of R is an overring S such that for each $a \in S$ there corresponds $r \in R$ such that $ar \in R$ and $ar \neq 0$. A theorem of R. E. Johnson [1] states that R possesses a right quotient ring S which is a (von Neumann) regular ring if and only if R has vanishing right singular ideal. In this case R possesses a unique (up to isomorphism over R) maximal right quotient ring S, and S is regular and right self-injective (Johnson-Wong [1]). It is easy to see that S is the injective hull of R, considering both rings as right R-modules in the natural way. Thus, each right ideal I of R has an injective hull $\hat{I}_R$ contained in S. In this notation, $S=\hat{R}_R$, and we use $\hat{R}$ to denote the maximal right quotient ring of R hereafter. By the results of Johnson [2], $\hat{I}_R$ can be characterized in two ways:

(a) $\hat{I}_R$ is the unique maximal essential extension of I contained in the right R-module $\hat{R}$.

(b) $\hat{I}_R$ is the principal right ideal of $\hat{R}$ generated by I.

Since $\hat{I}_R$ is therefore a right ideal of $\hat{R}$, $\Delta = \mathrm{Hom}_{\hat{R}}(\hat{I}_R, \hat{I}_R)$ is defined. Setting $\Gamma = \mathrm{Hom}_R(I, I)$, one of our main results (Theorem 2) states that $\hat{\Gamma} = \mathrm{Hom}_{\hat{R}}(\hat{I}_R, \hat{I}_R) = \Delta$. This means that Γ has vanishing right singular ideal, and that Δ is the maximal right quotient ring of Γ.

Since $\hat{I}_R$ is a principal right ideal in the regular ring $\hat{R}$, there exists an idempotent $e \in \hat{R}$ such that $\hat{I}_R = e\hat{R}$. Then, of course, $\Delta \cong e\hat{R}e$, and it is natural to investigate the relationship between $e\hat{R}e$ and $K = e\hat{R}e \cap R$. In general, it is too much to hope that $e\hat{R}e = \hat{K}$, since it is possible that $K=0$ for some nonzero $e \in \hat{R}$. Nevertheless, under the assumption that $\hat{R}$ is also a left quotient ring of R, or in case e is a primitive idempotent satisfying $e\hat{R}e \cap R \neq 0$, we establish (Theorem 3) that K has vanishing right singular ideal, and that $e\hat{R}e = \hat{K}$ (= the maximal right quotient ring of K.) In any case, for any nonzero idempotent $e \in \hat{R}$, $e\hat{R}e$ is the maximal right quotient ring of eRe.

Since we are not restricting ourselves to rings with identity, we say

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that an arbitrary (right) module M_R over a ring R is injective in case it possesses the following property: if A_R is any module, and if B_R is any submodule, then any homomorphism $B_R \to M_R$ can be extended to a homomorphism $A_R \to M_R$. In case R has an identity element, then a unital module M_R is u-injective in case it has the property above with A_R ranging over all unital modules. It is easy to see that a unital module M_R is injective if and only if it is u-injective (cf. Faith-Utumi [1]). *Baer's criterion* (loc. cit.) states that a unital module M_R is injective if and only if it has the following property: If f is any module homomorphism of a right ideal I of R into M, then there exists $m \in M$ such that $f(x) = mx \, \forall x \in I$. Call this latter property of a module M_R *Baer's condition*. It is known (loc. cit.) that if M_R is an arbitrary injective module, then it satisfies Baer's condition. Accordingly, if S is any ring which is right self-injective, the identity map $x \to x$ can be performed by a left multiplication by an element $e \in S$ which is patently a left identity element of S. If S is left-faithful, any left identity is two-sided. In particular, any semiprime right self-injective ring possesses an identity element. We use this fact below.

THEOREM 1. *If S is semiprime and right self-injective, then for any idempotent $e \in S$, eSe is semiprime and right self-injective.*

PROOF. Let I be any right ideal of eSe which is nilpotent of index 2. Then $(IS)^2 = (IS)(IS) = (IeS)(eIS) = [I(eSe)]IS \subseteq I^2S = 0$. Thus, IS is a nilpotent right ideal of S, whence $IS = 0$ and $I = 0$. Since eSe does not contain nilpotent right ideals of index 2, it follows that eSe is semiprime.

Now let I be any right ideal of eSe, let $x = \sum_1^n x_i s_i$, $x_i \in I$, $s_i \in S$, $i = 1, \cdots, n$ be any element of IS. Let $f \in \operatorname{Hom}_{eSe}(I, eSe)$, and let T denote the set of all elements $\sum_1^n f(x_i) r_i \in \sum_1^n f(x_i) S$, $r_i \in S$, $i = 1, \cdots, n$, such that $\sum_1^n x_i r_i = 0$. Clearly T is a right ideal of S, and $T \subseteq \sum_1^n f(x_i) S \subseteq eS$. Now if $t = \sum_1^n f(x_i) r_i \in T$, then

$$te = \left[\sum_1^n f(x_i) r_i \right] e = \sum_1^n [f(x_i)e] r_i e = \sum_1^n f(x_i)(e r_i e)$$
$$= \sum_1^n f(x_i e r_i e) = f\left(\sum_1^n x_i r_i e \right) = f(0) = 0.$$

Thus $T^2 = (eT)^2 = 0$, and $T = 0$, since S is semiprime. It follows that $x = \sum_1^n x_i s_i = 0$ implies that $\sum_1^n f(x_i) s_i = 0$, so that the correspondence

$$f' : x \to \sum_1^n f(x_i) s_i,$$

defined for any $x \in IS$, is an element of $\mathrm{Hom}_S(IS, S)$. Since S_S is injective, and S is semiprime, S has an identity element, so S_S satisfies Baer's condition. Accordingly, there exists $m \in S$ such that $f'(x) = mx \forall x \in IS$. In particular, if $x \in I$, then $x = xe$, so that $f'(x) = f(x)e = f(x)$. Thus, $f(x) = (eme)x \forall x \in I$. Since $eme \in eSe$, this shows that $(eSe)_{eSe}$ satisfies Baer's condition, and eSe is therefore right self-injective.

The proof of the next theorem needs some facts about essential and rational extensions, and the proofs of these may be found in Findlay-Lambek [1] and Johnson-Wong [1]. Recall that M_R is an *essential extension of a submodule* N_R in case each nonzero submodule of M_R has nonzero intersection with N_R; we symbolize this by $(M \nabla N)_R$; $_R(M \nabla N)$ denotes the right-left symmetry for a left module $_RM$. Then N_R is an *essential submodule of* M_R. An *essential left ideal* I *of* R is a left ideal such that $R \nabla I$.

The *singular submodule* $Z(M_R)$ is defined by:

$$Z(M_R) = \{x \in M \mid xI = 0 \text{ for some essential right ideal } I \text{ of } R\}.$$

We let $Z_r(R)$ denote $Z(R_R)$; it is an ideal, called the *right singular ideal of* R.

A module M_R is *rational* over a submodule N_R in case it has the following property: if P is any module satisfying $M \supseteq P \supseteq N$, and if $f \in \mathrm{Hom}_R(P, M)$, then $f = 0$ if and only if $f(N) = 0$. We let $(M \blacktriangledown N)_R$ denote a rational extension M of N. Any rational extension is essential; moreover:

If $Z(N_R) = 0$, then $(M \blacktriangledown N)_R$ if and only if $(M \nabla N)_R$.

We shall use the following characterization of rational extensions: $(M \blacktriangledown N)_R$ if and only if for each pair $x, y \in M$ with $y \neq 0$, there exist $r \in R$ and an integer n such that

$$xr + xn \in N \quad \text{and} \quad yr + yn \neq 0.$$

We also need the following facts about quotient rings (Johnson [2]): Let R be such that $Z_r(R) = 0$, and adopt the notation of the introduction. For each right ideal I of R, let $\bar{I} = \hat{I}_R \cap R$. Then $\bar{I}$ is the unique maximal essential extension of I_R contained in R; I is a *closed* right ideal of R in case $I = \bar{I}$. The totality $C_r(R)$ of closed right ideals of R is a complete lattice, and $C_r(\hat{R})$ is isomorphic $C_r(R)$ under the contraction map $A \to A \cap R$. From what we already have said, $C_r(\hat{R})$ consists of the principal right ideals of $\hat{R}$.

THEOREM 2. *Let R be a semiprime ring such that $Z_r(R) = 0$, let I be any right ideal of R, let $\hat{I}_R$ denote the principal right ideal of $\hat{R}$ generated by I, let $\Gamma = \mathrm{Hom}_R(I, I)$, and let $\Delta = \mathrm{Hom}_{\hat{R}}(\hat{I}_R, \hat{I}_R)$. Then $Z_r(\Gamma) = Z_r(\Delta) = 0$, and $\Delta = \hat{\Gamma}$ (= the maximal right quotient ring of Γ).*

PROOF. Set $S=\hat{R}$. As remarked above, $\hat{I}_R$ is the injective hull of I_R contained in S_R, and is the principal right ideal of S generated by I. Since S is regular, $\hat{I}_R=eS$, where $e=e^2\in S$.

We first show that $\Delta=\mathrm{Hom}_S(\hat{I}_R, \hat{I}_R)$ coincides with $\Omega=\mathrm{Hom}_R(\hat{I}, \hat{I})$. Clearly $\Omega\supseteq\Delta$. Conversely if $f\in\Omega$, and if $r\in\hat{I}_R\cap R=\bar{I}$, then $f(r)=f(e)r$. If $x\in\hat{I}_R$, then $x_R=\{t\in R\mid xt\in I\}$ is an essential right ideal of R. Now if $t\in x_R$, then

$$f(x)t = f(xt) = f(e)xt,$$

that is, $(f(x)-f(e)x)t=0$. Thus, $[f(x)-f(e)x]x_R=0$. Since $\hat{I}_R$ has zero singular submodule, we conclude that $f(x)=f(e)x\,\forall x\in\hat{I}_R$, and then clearly $f\in\Delta$. Thus establishes $\Omega=\Delta$.

Since $\hat{I}_R$ is the injective hull of I_R, it follows that each $\gamma\in\Gamma$ has an extension $\hat{\gamma}\in\Delta=\Omega$, and $\hat{\gamma}$ is unique, since $\hat{I}_R$ is rational over I_R. Clearly $\{\hat{\gamma}\in\Delta\mid\gamma\in\Gamma\}$ is a subring of Δ isomorphic to Γ under $\gamma\leftrightarrow\hat{\gamma}$. Henceforth, consider Γ as a subring of Δ.

Now Δ is isomorphic to the ring eSe. If I_L denotes the totality of left multiplications a_L of I by elements $a\in I$,

$$a_L: x\rightarrow ax \qquad x\in I,$$

then I_L is a subring of Γ, and the natural isomorphism $\Delta\cong eSe$ maps I_L onto eIe and maps Γ onto a subring Γ_e of eSe. Since $\Gamma_e\supseteq eIe$, in order to show that $(eSe\,\nabla\Gamma_e)_{\Gamma_e}$, it suffices to show that $(eSe\,\nabla eIe)_{eIe}$.

Now let $0\neq\delta\in eSe$. Since $(eS\,\nabla I)_R$, there exists $r\in R$ such that $0\neq\delta r\in I$. Since $\delta r=\delta(er)$, it follows that $er\neq 0$. By the same reasoning, since $(eS\,\nabla I)_R$, there exists $s\in R$, $n\in\mathbf{Z}$, such that $u=er(s+n)\in I$, and $w=\delta r(s+n)\neq 0$. Since $\delta r\in I$, it follows that $w\in I$. Since R is semiprime, R is left-faithful, hence $wR\neq 0$ and also $(wR)^2\neq 0$. Therefore, one can choose $t\in R$ such that $w'=wt$ satisfies $w''=w'e\neq 0$. Then $w'=\delta u'$, where $u'=ut\in I$ and $w'\in I$, and

$$0\neq w''=\delta u'',$$

with $u''=u'e\in eIe$ and $w''\in eIe$. Thus, $(eSe\,\nabla eIe)_{eIe}$ as asserted. Hence eSe is a right quotient ring of Γ_e, and Δ is a right quotient ring of Γ. Now S is regular (hence semiprime), so that $\Delta=eSe$ is right self-injective by Theorem 1. Thus, Δ is a maximal right quotient ring of Γ. Since $\Delta(\cong eSe)$ is regular, $Z_r(\Gamma)=Z_r(\Delta)=0$.

THEOREM 3. *Let R be any semiprime ring satisfying $Z_r(R)=0$, and let e be any idempotent in $S=\hat{R}$. Then*:

(1) *$eSe=\hat{K}$, where $K=eRe$.*

(2) *If S is also a left quotientring of R, then $eSe=\hat{K}$, where $K=eSe\cap R$, and eSe is also a left quotient ring of K.*

(3) *If e is a primitive idempotent, and if $eSe\cap R\neq 0$, then eSe is the right quotient field of $K=eSe\cap R$.*

Proof. Let $B=(eS\cap R)+(1-e)S\cap R$. The lattice isomorphism $C_r(\hat{R})\cong C_r(R)$ implies that

$$\lfloor eS \nabla eS \cap R\rfloor_R \text{ (resp. } \lfloor(1-e)S\nabla(1-e)S\cap R\rfloor_R),$$

and it follows that $(S\nabla B)_R$. For each $x\in S$, $x_B=\{b\in B\mid xb\in B\}$ is a right ideal of B, and $(R\nabla x_B)_R$.

Now choose $\delta\in eSe$ and $\delta\neq 0$. Since δ_B is an essential right ideal of R, then $\delta\delta_B\neq 0$ (since $Z(S_R)=Z(R_R)=0$). By semiprimeness of R, we see that $(\delta\delta_B)^2\neq 0$, so that $\delta\delta_B e\neq 0$. Hence we can choose $b\in\delta_B\subseteq B$ such that $\delta be\neq 0$. (Note that $eb\in eS\cap R$.)

Case (1). Now $\delta b\in R$, and $b\in R$, and $\delta be=\delta(ebe)=e(\delta b)e$. Thus $0\neq\delta(ebe)\in eRe$ with $ebe\in eRe$. This shows that eSe is a right quotient ring of eRe.

Case (2). Since S is a regular ring which is both a right and left quotient ring of R, necessarily $Z_l(R)=0$. Then ${}_RS$ is a rational extension of ${}_RR$, and, moreover, ${}_R(Se)$ is a rational extension of ${}_R(Se\cap R)$. Now the correspondence $x\rightarrow x\delta be \forall x\in Se$ is an element $f\in\mathrm{Hom}_R(Se,\ Se)$, and $f\neq 0$ since $e(\delta be)=\delta be\neq 0$. It follows that $f(Se\cap R)\neq 0$, that is, that $(Se\cap R)\delta be\neq 0$ and $(Se\cap R)\delta b\neq 0$. By the semiprimeness of R, $[(Se\cap R)\delta b]^2\neq 0$, and so $\delta b(Se\cap R)\neq 0$. Hence choose $u\in Se\cap R$ such that $\delta bu\neq 0$. Since b, δb, $u\in R$, then also δbu, $bu\in R$, and

$$\delta bu = (e\delta)b(ue) = e(\delta bu)e \in eSe\cap R = K.$$

Since $eb\in eS\cap R$, and $u\in Se\cap R$, then $k=ebu\in K$, so that $0\neq\delta k=\delta bu\in K$, with $k\in K$. Since δ was an arbitrary nonzero element of eSe, this proves that eSe is a right quotient ring of K.

Case (3). eSe is a division ring and Se is a right vector space over eSe. Since $Se\cap R\neq 0$, and since $0\neq\delta be\in eSe$, it follows that $(Se\cap R)\delta be\neq 0$, and the rest of the proof proceeds as in the proof of (2).

In all cases we have deduced that eSe is a right quotient ring of K without resource to the fact that the right quotient ring S of R is maximal. Now in Case (2), $Z_l(R)=0$ is a consequence of the fact that S is a regular ring which is a left quotient ring of R. Hence, by symmetry, we conclude that eSe is a left quotient ring of K in this case.

Since eSe is regular along with S, and since eSe is right self-injective by Theorem 1, we conclude that $eSe=\hat{K}$ in all cases, completing the proof.

Remark. It can be shown that K in (2) need not be semiprime.

We construct an example which shows that (2) and (3) of Theorem 3 fail under a weakening of the hypothesis.

Let K be a right Ore domain which is not a left Ore domain, and let k_1, k_2 be nonzero elements of K such that $Kk_1 \cap Kk_2 = 0$.

If D denotes the right quotient division ring of K, then $S = D_2$, the full ring of all 2×2 matrices over D, is the classical, and maximal, right quotient ring of $R = K_2$. Let $\{e_{ij} \mid i, j = 1, Z\}$ denote matrix units in S, let $a = k_1^{-1}e_{11} + k_2^{-1}e_{12}$, and suppose $b \in S$ is such that $ba \in R$. Then $b = \sum_{i,j=1}^{2} c_{ij}e_{ij}$, with $c_{ij} \in D$, $i, j = 1, 2$, and

$$ba = c_{11}k_1^{-1}e_{11} + c_{11}k_2^{-1}e_{12} + c_{21}k_1^{-1}e_{21} + c_{21}k_2^{-1}e_{22}.$$

Since $ba \in K_2 = R$, necessarily

$$c_{11}k_1^{-1},\ c_{11}k_2^{-1},\ c_{21}k_1^{-1},\ c_{21}k_2^{-1} \in K,$$

and then c_{11}, $c_{21} \in Kk_1 \cap Kk_2$. Since $Kk_1 \cap Kk_2 = 0$, $c_{11} = c_{22} = 0$, so necessarily $ba = 0$. This shows that $Sa \cap R = 0$. Now $e = k_1 a = e_{11} + k_1 k_2^{-1} e_{12}$ belongs to Sa, and $e = e^2$. It is easy to see that eSe is a division ring (or equivalently, that Se is a minimal left ideal of S), while $eSe \cap R = 0$. In particular, eSe is not a quotient ring of $eSe \cap R$.

In view of Theorem 3, it is of interest to consider conditions which imply that the maximal right quotient ring is also a left quotient ring. The general question has been extensively treated by Utumi [1].

References

C. Faith and Y. Utumi

1. *Quasi-injective modules and their endomorphism rings*, Arch. Math. **15** (1964), 166–174.

2. *On noetherian prime rings*, Trans. Amer. Math. Soc. **114** (1965), 53–60.

G. Findlay and J. Lambek

1. *A generalized ring of quotients*. I, Canad. Math. Bull. **1** (1958), 77–85.

N. Jacobson

1. *Structure of rings*, Amer. Math. Soc. Colloq. Publ. Vol. 37, Amer. Math. Soc., Providence, R. I., 1956.

R. E. Johnson

1. *The extended centralizer of a ring over a module*, Proc. Amer. Math. Soc. **2** (1951), 891–895.

2. *Quotient rings of rings with zero singular ideal*, Pacific J. Math. **11** (1961), 1385–1392.

R. E. Johnson and E. T. Wong

1. *Self-injective rings*, Canad. Math. Bull. **2** (1959), 167–173.

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1. *On rings of which any one-sided quotient rings are two-sided*, Proc. Amer. Math. Soc. **14** (1963), 141–147.

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