REFLEXIVE SEMIGROUPS

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By a character of a semigroup $S$ we mean a homomorphism $\chi$ of $S$ into the multiplicative semigroup of complex numbers with the property that $\chi(1) \neq 0$ if $S$ has an identity element 1. We denote by $S^*$ the semigroup of all characters of $S$ with respect to pointwise multiplication, and $S^{**}$ is the character semigroup of $S^*$. In [3] Warne and Williams raised the question: when are $S$ and $S^{**}$ isomorphic? There is, of course, always a natural homomorphism of $S$ into $S^{**}$ and we interpret their question to mean: when are $S$ and $S^{**}$ naturally isomorphic? The purpose of this note is to answer the latter question. The natural homomorphism that we have referred to is the mapping $s \mapsto \psi_s$, where $\psi_s(\chi) = \chi(s)$ for each $\chi$ in $S^*$.

Definition. A semigroup $S$ is said to be reflexive if $S$ and $S^{**}$ are naturally isomorphic, that is, if $s \mapsto \psi_s$ is an isomorphism from $S$ onto $S^{**}$.

It is well known (see, for example, [1]) that a commutative semigroup $S$ is an inverse semigroup if and only if $S$ is a semilattice of groups.

Lemma 1. For any semigroup $S$, the semigroup $S^*$ is a commutative inverse semigroup with identity.

Proof. These properties of $S^*$ are inherited naturally from the multiplicative semigroup of complex numbers.

Corollary 1. Any reflexive semigroup is a commutative inverse semigroup with identity.

Theorem 1. A semigroup $S$ is reflexive if and only if $S$ is a reflexive semilattice of reflexive groups.

Proof. We may assume that $S$ is a semilattice $E$ of commutative groups $\{G_e\}_{e\in E}$ where $E$ is the set of idempotent elements of $S$. Let $f$ denote the restriction to $E$ of the natural mapping of $S$ into $S^{**}$. Let $g$ denote the natural mapping of $E$ into $E^{**}$. Define $h$ from $E^{**}$ into $S^{**}$ by: $h(\theta) = \pi$ where $\pi(\chi) = \theta(\chi \mid E)$. Then $h$ is an isomorphism from $E^{**}$ onto the set $F$ of idempotents of $S^{**}$ and $f = h \cdot g$. Thus $f$ is an isomorphism from $E$ onto $F$ if and only if $g$ is an isomorphism from $E$ onto $E^{**}$, that is, if and only if $E$ is reflexive.

Presented to the Society, August 25, 1964; received by the editors July 8, 1964.
For each \( e \in E \) the diagram

\[
\begin{array}{c}
G_e \xrightarrow{f} S^{**} \\
\downarrow g \quad \downarrow h \\
G_e^{**}
\end{array}
\]

is commutative where: \( f \) is the restriction to \( G_e \) of the natural map from \( S \) into \( S^{**} \), \( g \) is the natural map from \( G_e \) into \( G_e^{**} \), and \( h(\emptyset) = \pi \) where

\[
\pi(\chi) = \begin{cases} 
\theta(\chi) | G_e & \text{if } \chi(\emptyset) \neq 0, \\
0 & \text{if } \chi(\emptyset) = 0.
\end{cases}
\]

Moreover, \( h \) is an isomorphism from \( G_e^{**} \) onto \( H_{f(\emptyset)} \) where \( H_{f(\emptyset)} \) is the maximal subgroup of \( S^{**} \) having \( f(\emptyset) \) as an identity element. Hence \( f \) is an isomorphism from \( G_e \) onto \( H_{f(\emptyset)} \) if and only if \( G_e \) is reflexive. The theorem now follows.

**Lemma 2.** Suppose that \( E \) is a commutative idempotent semigroup. Necessary and sufficient conditions that \( E \) be reflexive are that \( E \) have an identity element and that every nonvoid subset of \( E \) contain a minimal element and a maximal element.

**Proof.** The sufficiency was shown in [3]. It follows from Corollary 1 that it is necessary for \( E \) to have an identity element.

Assume that there is an infinite descending chain in \( E \),

\[
e_1 > e_2 > \cdots > e_n > \cdots
\]

Define

\[
E^*_i = \{ \chi \in E^* \mid \chi(e_i) = 1 \text{ for } i = 1, 2, 3, \cdots \}.
\]

The complement of \( E^*_i \) in \( E^* \) is a prime ideal of \( E^* \). Thus the characteristic function \( \pi \) of the subset \( E^*_i \) of \( E^* \) is a character of \( E^* \), so \( \pi \) is an element of \( E^{**} \). Suppose that there is an element \( e \in E \) such that \( \chi(e) = \pi(\chi) \) for each \( \chi \in E^* \). Then for each \( \chi \in E^* \), we have that \( \chi(e) = 1 \) if and only if \( \chi(e_i) = 1 \) for each \( i \). However, it is easy to show that this is not the case by considering the characteristic functions of the following subsets of \( E \):

\[
E_0 = \{ x \in E \mid x \geq e_i \text{ for some } i \},
\]
\[
E_i = \{ x \in E \mid x \geq e_i \text{ for } i \geq 1 \}.
\]

We conclude that there is no element \( e \in E \) which maps \( \pi \) under
the natural map and that \( E \) is not reflexive if \( E \) contains an infinite descending chain.

A rather similar argument shows that if \( E \) is reflexive then \( E \) cannot contain an infinite ascending chain

\[
e_1 < e_2 < \cdots < e_n < \cdots
\]

In this case, we would define

\[
E^*_1 = \{ \chi \in E^* | \chi(e_i) = 1 \text{ for all but a finite number of } i \}
\]

and observe that no element of \( E \) maps onto the characteristic function \( \pi \) of \( E^*_1 \). Thus for \( E \) to be reflexive it is necessary for each non-void subset of \( E \) to contain a minimal element and a maximal element.

**Corollary 2.** A semilattice is reflexive if and only if it is a (complete) lattice with no infinite chain.

The following lemma follows from the results of [2].

**Lemma 3.** A group is reflexive if and only if it is finite and commutative.

We have now proved the following

**Theorem 2.** A semigroup is reflexive if and only if it is a lattice \( L \) of finite commutative groups where \( L \) has no infinite chain.

**References**


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