REFLEXIVE SEMIGROUPS

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By a character of a semigroup S we mean a homomorphism χ of S into the multiplicative semigroup of complex numbers with the property that $\chi(1)\neq 0$ if S has an identity element 1. We denote by S^* the semigroup of all characters of S with respect to pointwise multiplication, and S^{**} is the character semigroup of S^*. In [3] Warne and Williams raised the question: when are S and S^{**} isomorphic? There is, of course, always a natural homomorphism of S into S^{**} and we interpret their question to mean: when are S and S^{**} naturally isomorphic? The purpose of this note is to answer the latter question. The natural homomorphism that we have referred to is the mapping $s\to\psi_s$ where $\psi_s(\chi)=\chi(s)$ for each χ in S^*.

Definition. A semigroup S is said to be *reflexive* if S and S^{**} are naturally isomorphic, that is, if $s\to\psi_s$ is an isomorphism from S onto S^{**}.

It is well known (see, for example, [1]) that a commutative semigroup S is an inverse semigroup if and only if S is a semilattice of groups.

Lemma 1. *For any semigroup S, the semigroup S^* is a commutative inverse semigroup with identity.*

Proof. These properties of S^* are inherited naturally from the multiplicative semigroup of complex numbers.

Corollary 1. *Any reflexive semigroup is a commutative inverse semigroup with identity.*

Theorem 1. *A semigroup S is reflexive if and only if S is a reflexive semilattice of reflexive groups.*

Proof. We may assume that S is a semilattice E of commutative groups $\{G_e\}_{e\in E}$ where E is the set of idempotent elements of S. Let f denote the restriction to E of the natural mapping of S into S^{**}. Let g denote the natural mapping of E into E^{**}. Define h from E^{**} into S^{**} by: $h(\theta)=\pi$ where $\pi(\chi)=\theta(\chi|E)$. Then h is an isomorphism from E^{**} onto the set F of idempotents of S^{**} and $f=h\cdot g$. Thus f is an isomorphism from E onto F if and only if g is an isomorphism from E onto E^{**}, that is, if and only if E is reflexive.

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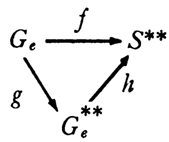
For each $e \in E$ the diagram



is commutative where: f is the restriction to G_e of the natural map from S into S^{**}, g is the natural map from G_e into G_e^{**}, and $h(\theta) = \pi$ where

$$\pi(\chi) = \begin{cases} \theta(\chi \mid G_e) & \text{if } \chi(e) \neq 0, \\ 0 & \text{if } \chi(e) = 0. \end{cases}$$

Moreover, h is an isomorphism from G_e^{**} onto $H_{f(e)}$ where $H_{f(e)}$ is the maximal subgroup of S^{**} having $f(e)$ as an identity element. Hence f is an isomorphism from G_e onto $H_{f(e)}$ if and only if G_e is reflexive. The theorem now follows.

LEMMA 2. *Suppose that E is a commutative idempotent semigroup. Necessary and sufficient conditions that E be reflexive are that E have an identity element and that every nonvoid subset of E contain a minimal element and a maximal element.*

PROOF. The sufficiency was shown in [3]. It follows from Corollary 1 that it is necessary for E to have an identity element.

Assume that there is an infinite descending chain in E,

$$e_1 > e_2 > \cdots > e_n > \cdots.$$

Define

$$E_1^* = \{\chi \in E^* \mid \chi(e_i) = 1 \text{ for } i = 1, 2, 3, \cdots\}.$$

The complement of E_1^* in E^* is a prime ideal of E^*. Thus the characteristic function π of the subset E_1^* of E^* is a character of E^*, so π is an element of E^{**}. Suppose that there is an element $e \in E$ such that $\chi(e) = \pi(\chi)$ for each $\chi \in E^*$. Then for each $\chi \in E^*$, we have that $\chi(e) = 1$ if and only if $\chi(e_i) = 1$ for each i. However, it is easy to show that this is not the case by considering the characteristic functions of the following subsets of E:

$$E_0 = \{x \in E \mid x \geqq e_i \text{ for some } i\},$$
$$E_i = \{x \in E \mid x \geqq e_i \text{ for } i \geqq 1\}.$$

We conclude that there is no element $e \in E$ which maps π under the natural map and that E is not reflexive if E contains an infinite descending chain.

A rather similar argument shows that if E is reflexive then E cannot contain an infinite ascending chain

$$e_1 < e_2 < \cdots < e_n < \cdots .$$

In this case, we would define

$$E_1^* = \{\chi \in E^* \mid \chi(e_i) = 1 \text{ for all but a finite number of } i\}$$

and observe that no element of E maps onto the characteristic function π of E_1^*. Thus for E to be reflexive it is necessary for each nonvoid subset of E to contain a minimal element and a maximal element.

COROLLARY 2. *A semilattice is reflexive if and only if it is a (complete) lattice with no infinite chain.*

The following lemma follows from the results of [2].

LEMMA 3. *A group is reflexive if and only if it is finite and commutative.*

We have now proved the following

THEOREM 2. *A semigroup is reflexive if and only if it is a lattice L of finite commutative groups where L has no infinite chain.*

REFERENCES

1. A. H. Clifford and G. B. Preston, *Algebraic theory of semigroups*, Math. Surveys No. 7, Amer. Math. Soc., Providence, R. I., 1961.

2. L. Fuchs, *On character groups of discrete abelian groups*, Acta. Math. Acad. Sci. Hung. 10 (1959), 133–140.

3. R. J. Warne and L. K. Williams, *Characters on inverse semigroups*, Czechoslovak Math. J. 11 (1961), 150–155.

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