INEQUALITIES AND A LIMIT THEOREM FOR CERTAIN WIENER INTEGRALS

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The Wiener space C_w of real-valued continuous functions $x(t)$ on $[0, 1]$ with $x(0) = 0$ is an inner product space with respect to the inner product

$$\langle x \mid y \rangle = \int_0^1 x(t)y(t)\,dt, \qquad x, y \in C_w.$$

Let $\|x\|$, $x \in C_w$, be the associated Hilbert norm. According to Cameron and Martin [1],

$$\int_{C_w} \exp\{\lambda \|x\|^2\}\, d_w x = (\sec \lambda^{1/2})^{1/2}, \qquad 0 \leqq \lambda < \frac{\pi^2}{4} \cdot \tag{0}$$

In the present paper we consider

$$\int_{C_w} \exp\{\lambda \langle x \mid y \rangle\}\, d_w x.$$

Our result is the following

THEOREM 1. *Let $f(u)$ be a real-valued Lebesgue measurable function such that $\int_{-\infty}^{\infty} f(u)e^{-u^2}\,du$ converges absolutely as a Lebesgue integral. Then for any $y \in L_2[0, 1]$ and complex number μ*

$$\begin{aligned} &\int_{C_w} f[x(1)] \exp\{\mu \langle x \mid y \rangle\}\, d_w x \\ &\quad = \frac{1}{\pi^{1/2}} \exp\left\{\frac{\mu^2}{4}\left(\|Y\|^2 - [Z(1)]^2\right)\right\} \\ &\qquad \cdot \int_{-\infty}^{\infty} f(u) \exp\{-u^2 + \mu[Y(1) - Z(1)]u\}\, du \end{aligned} \tag{1}$$

where

$$Y(t) = \int_0^t y(s)\,ds, \qquad Z(t) = \int_0^t Y(s)\,ds, \qquad t \in [0, 1]. \tag{2}$$

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2. *In particular*

$$\int_{C_w} \exp\{\mu\langle x \mid y\rangle\}\, d_w x = \exp\left\{\frac{\mu^2}{4}(\|Y\|^2 - 2Z(1)Y(1) + [Y(1)]^2)\right\}. \tag{3}$$

3. *Furthermore for any real* α,

$$\exp\left\{\frac{\alpha^2}{4}\langle x_0 \mid y\rangle^2\right\} \leqq \int_{C_w} \exp\{\alpha\langle x \mid y\rangle\}\, d_w x \leqq \exp\{\alpha^2\|y\|^2\}, \tag{4}$$

$$\exp\{-\alpha^2\|y\|^2\} \leqq \int_{C_w} \exp\{i\alpha\langle x \mid y\rangle\}\, d_w x \leqq \exp\left\{-\frac{\alpha^2}{4}\langle x_0 \mid y\rangle^2\right\} \tag{5}$$

where $x_0(t) = t$ *on* $[0, 1]$.

COROLLARY 1. *For* $0 \leqq \alpha < \pi/2$

$$\left|\int_{C_w \times C_w} \exp\{\alpha\langle x \mid y\rangle\}(d_w x) \times (d_w y)\right| \leqq (\sec \alpha)^{1/2} \tag{6}$$

where the integral is a double Wiener integral on $C_w \times C_w$ *with respect to the product measure.*

COROLLARY 2. *Let* α *be real. Then for any* $\{y_n\} \subset C_w$ *such that* $\lim_{n\to\infty} |\langle x_0 | y_n\rangle| = \infty$,

$$\lim_{n\to\infty} \int_{C_w} \exp\{i\alpha\langle x \mid y_n\rangle\}\, d_w x = 0. \tag{7}$$

PROOF OF THE THEOREM. For $y \in C_w$, $Y(t)$ as defined by (2) is of bounded variation and furthermore $Y'(t) = y(t)$ almost everywhere on $[0, 1]$. Thus for any $x \in C_w$, by integration by parts

$$\int_0^1 x(t)y(t)\, dt = \int_0^1 x(t)\, dY(t) = Y(1)\int_0^1 dx(t) - \int_0^1 Y(t)\, dx(t). \tag{8}$$

If $y(t) \equiv 0$ almost everywhere on $[0, 1]$, $\langle x | y\rangle = 0$ for all $x \in C_w$ and (1) holds trivially. Assume that $y(t)$ is not almost identically vanishing on $[0, 1]$. Then $Y(t)$ is not constant on $[0, 1]$ so that the function which is identically equal to 1 on $[0, 1]$ and Y are linearly independent on $[0, 1]$. By the Gram-Schmidt procedure we obtain an equivalent orthonormal system consisting of the function identically equal to 1 and

$$W(t) = \frac{1}{(\|Y\|^2 - [Z(1)]^2)^{1/2}}\{Y(t) - Z(1)\} \tag{9}$$

where $Z(t)$ is as defined by (2). Now solving (9) for $Y(t)$ and substituting the result in (8) we have

$$\int_0^1 x(t)y(t)\,dt = c_1(y)\int_0^1 dx(t) + c_2(y)\int_0^1 W(t)\,dx(t) \tag{10}$$

where

$$c_1(y) = Y(1) - Z(1), \qquad c_2(y) = -\,(\|Y\|^2 - [Z(1)]^2)^{1/2} \tag{11}$$

According to Paley-Wiener [2], if $\{\alpha_k(t)\}$ is an orthonormal set of real valued functions of bounded variation on $[0, 1]$,

$$\int_{C_w} \phi\left[\int_0^1 \alpha_1(t)\,dx(t), \cdots, \int_0^1 \alpha_n(t)dx(t)\right]d_wx$$
$$= \frac{1}{\pi^{n/2}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\phi(u_1, \cdots, u_n)$$
$$\cdot\exp\{-(u_1^2 + \cdots + u_n^2)\}\,du_1\cdots du_n$$

for every Lebesgue measurable function $\phi(u_1, \cdots, u_n)$ for which the integral on the right side converges absolutely as a Lebesgue integral. Applying this result together with (10) and

$$\int_{-\infty}^{\infty}\exp\{-s^2 + as\}\,ds = \pi^{1/2}\exp\left\{\frac{a^2}{4}\right\},$$

we have

$$\int_{C_w} f[x(1)]\exp\{\mu\langle x \mid y\rangle\}\,d_wx$$
$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\exp\{\mu c_1(y)u - u^2\}\,du\cdot\int_{-\infty}^{\infty}\exp\{\mu c_2(y)v - v^2\}\,dv$$
$$= \frac{1}{\pi^{1/2}}\exp\left\{\frac{\mu^2}{4}[c_2(y)]^2\right\}\int_{-\infty}^{\infty} f(u)\exp\{\mu c_1(y)u - u^2\}\,du,$$

which is (1).

Choosing $f(u)\equiv 1$, we have

$$\int_{C_w}\exp\{\mu\langle x \mid y\rangle\}\,d_wx = \exp\left\{\frac{\mu^2}{4}[c_2(y)]^2\right\}\exp\left\{\frac{\mu^2}{4}[c_1(y)]^2\right\}$$
$$= \exp\left\{\frac{\mu^2}{4}(\|Y\|^2 - 2Z(1)Y(1) + [Y(1)]^2)\right\}$$

according to (11). This proves (3).

To prove (4), (5) we remark that according to Schwarz's inequality, $|Z(1)| \leqq \|Y\|$ so that

$$\begin{aligned}\{Z(1) - Y(1)\}^2 &\leqq \|Y\|^2 - 2Z(1)Y(1) + [Y(1)]^2 \\ &\leqq \{\|Y\| + |Y(1)|\}^2.\end{aligned} \tag{12}$$

Again by Schwarz's inequality we have $|Y(1)| \leqq \|y\|$ and also

$$\|Y\|^2 = \int_0^1 |Y(t)|^2 dt = \int_0^1 \left| \int_0^t y(s) ds \right|^2 dt \leqq \|y\|^2$$

so that

$$\{\|Y\| + |Y(1)|\}^2 \leqq 4\|y\|^2. \tag{13}$$

On the other hand

$$Z(1) = \int_0^1 Y(t)\, dt = \int_0^1 \left\{ \int_0^t y(s)\, ds \right\} dt = \int_0^1 y(s)(1 - s)\, ds$$

so that

$$Z(1) - Y(1) = -\int_0^1 sy(s)\, ds = -\langle x_0 | y \rangle \tag{14}$$

where $x_0(t) = t$ on $[0, 1]$. Thus combining (12), (13), (14) we have

$$\langle x_0 | y \rangle^2 \leqq \|Y\|^2 - 2Z(1)Y(1) + [Y(1)]^2 \leqq 4\|y\|^2. \tag{15}$$

Now (4), (5) follow from (3), (15). This completes the proof of the theorem.

Corollary 1 follows from (4) by means of Fubini's Theorem and (0). Corollary 2 follows from (5).

Bibliography

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