AN APPROXIMATION THEOREM FOR A CLASS OF OPERATORS

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It is well known that if $U$ is a unitary operator in a Hilbert space $H$, then the following approximation theorem is an immediate consequence of the spectral representation for the operator $U$.

**Theorem A.** (a) Let $\{E_t: 0 \leq t \leq 2\pi\}$ denote the family of spectral projections associated with $U$; then if $\epsilon > 0$ and $\alpha, 0 \leq \alpha \leq 2\pi$, are given and if $x$ is any element in the range of the projection $E_{\alpha+\epsilon} - E_{\alpha}$, we have:

$$\|(U - e^{i\alpha})x\| \leq \epsilon \|x\|;$$

(b) moreover, for this same $\epsilon$, there exists a finite collection of closed linear manifolds in $H$ such that $H$ is the direct sum of these manifolds; and in each subspace, $U$ behaves as in part (a).

E. R. Lorch [7] extended this theorem to certain classes of operators in a reflexive Banach space. The class considered by Lorch consists of those bounded, invertible operators $V$ for which the norms of their iterates are restricted by the condition, $\|V^n\| = O(1)$ as $|n|$ tends to infinity.

The author [6] extended the results of Lorch to a somewhat larger class restricted by the condition, $\|V^n\| = O(|n|)$. This extension was made through the use of methods developed by N. Dunford [4], [5].

In the present paper, the result is extended to a much larger class of operators by using the methods of Harmonic analysis as developed by A. Beurling [1], J. Wermer [8], Y. Domar [3], and others. It should be mentioned that this extension might have been possible through the use of methods developed by F. Wolf [9] in his spectral theory for operators based on the generalized trigonometric integrals of S. Bochner. The present class is restricted by the condition,

(i) $\|V^n\| = O(|n|^q)$ as $|n|$ tends to infinity, for some $q > 0$.

An operator $V$, defined in a Banach space $B$, which satisfies condition (i) is easily seen to have it spectrum on the circumference of the unit circle. Moreover, the usual operational calculus may be extended by introducing a certain weighted algebra associated with the sequence $\{\|V^n\|: n = 0, \pm 1, \cdots \}$. Such algebras were introduced by Beurling [1], and later generalized by Wermer and Domar.

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**Definition 1.** Let $p_n$ and $d_n$, $n = 0, \pm 1, \pm 2, \cdots$ be sequences of positive numbers such that $p_n \geq 1$, $p_{n+m} \geq p_n p_m$, and $p_n \leq d_n$, where $d_n$ is an even, increasing sequence. Denote by $A_\rho$ the space of all continuous, periodic functions $f(\theta)$ whose Fourier coefficients $f_n$ satisfy

$$\sum_{n=-\infty}^{\infty} |p_n f_n| < \infty.$$

When given the norm $||f|| = \sum_{n=-\infty}^{\infty} |p_n f_n|$, the space $A_\rho$ becomes a commutative Banach algebra under pointwise multiplication. Moreover, if $\rho_n$ is further restricted by $\sum_{n=-\infty}^{\infty} \log p_n/(1+n^2) < \infty$, the algebra $A_\rho$ is regular. In the present application, we take $p_n = ||V^n||$; the relevant conditions satisfied by $\rho_n$ follow directly from the restriction of polynomial growth on $||V^n||$.

Using the algebra $A_\rho$, an operational calculus may be generated for $V$ by setting $f(V) = \sum_{n=-\infty}^{\infty} f_n V^n$, for such $f$ in $A_\rho$. In order to apply the methods of harmonic analysis, we shall drop down to the space $B$ and look at the induced map on $A_\rho \times B \rightarrow B$ defined by $f \circ a = f(V)a$, where $f$ belongs to $A_\rho$ and $a$ to $B$. The idea of considering the induced map is an adaptation of a technique developed by Domar in his study of function algebras over locally compact Abelian groups.

Using the continuity of the mapping $(f, a) \rightarrow f \circ a$, we may associate with each element $a$ in $B$ a closed subset $\Lambda(a)$ of the circumference of the unit circle defined as the hull of the ideal $I(a) = \{f \in A_\rho : f \circ a = 0\}$. In addition, for each $f$ in $A_\rho$, let $\Lambda_f$ denote the closure of the support of $f$; then the following results are basic (cf. [3]).

**Lemma 1.** (a) $\Lambda(a)$ is void if and only if $a = 0$.
(b) For any $f$ in $A_\rho$ and $a$ in $B$, $\Lambda(f \circ a) \subseteq \Lambda_f \cap \Lambda(a)$.
(c) If $f$ and $g$ belong to $A_\rho$ and $f = g$ in some neighborhood of $\Lambda(a)$, then $f \circ a = g \circ a$.

The elements $\phi$ of the adjoint space $A_\rho^*$ may be identified with the space of sequences $\{\phi_n\}$ for which $\sup_n |\phi_n|/\rho_n < \infty$ by means of the representation $\phi(f) = \sum_{n=-\infty}^{\infty} f_n \phi_n$; furthermore $||\phi|| = \sup_n |\phi_n|/\rho_n$. With each $\phi$ in $A_\rho^*$, there is associated a pair of functions $\Phi^+(z) = \sum_{n=0}^{\infty} \phi_n z^{-n}$, $\Phi^-(z) = - \sum_{n=0}^{\infty} \phi_{-n} z^n$, where $\Phi^+$ and $\Phi^-$ are defined and analytic for $|z| > 1$ and $|z| < 1$, respectively. Using the pair $(\Phi^+, \Phi^-)$ we may define a closed set $\sigma(\phi)$ as the set of points $\lambda$ on the circumference of the unit circle for which the pair $(\Phi^+, \Phi^-)$ do not continue each other across any arc containing the point $\lambda$ (cf. [2], [8]).

On the other hand we may construct another representation of $A_\rho$, this time as an operator over the Banach space $A_\rho^*$ in place of the space $B$. In this case the action of $f$ as an operator in $A_\rho^*$ will be defined by $(f \circ \phi)(g) = \phi(fg)$. With this representation the set $\Lambda(\phi)$ may be defined just as was done for the elements of $B$. In [3] it was shown.
that \( \sigma(\phi) = \Lambda(\phi) \). Now for any \( a \in B \) and \( a^* \in B^* \), the sequence \( \phi_n = a^* (V^{n-1} a) \) defines an element \( \phi \) in \( A^* \). Using the definition of \( \sigma(\phi) \) we are led to the usual definition (cf. [5]) of the spectrum \( \sigma(a) \) of an element \( a \) in \( B \) as the set of points \( \lambda, |\lambda| = 1 \), such that for some \( a^* \) in \( B^* \), \( \lambda \) belongs to the spectrum of \( \{a^* (V^{n-1} a)\} \) as an element of \( A^* \).

**Theorem 1.** For each \( a \) in \( B \), \( \sigma(a) = \Lambda(a) \).

This result is established as follows. Let \( a \) be a fixed element in \( B \), and let \( \phi_{a^*} \) denote the element in \( A^* \) corresponding to the sequence \( \{a^* (V^n a)\} \). Observe that since \( (fg) \circ a = g \circ (f \circ a) \), we have

\[
\bigcap \{ I(\phi_{a^*}) : a^* \in A^* \} = \{ f \in A^* : a^* \circ g = 0 \text{ for all } g \in A, a^* \in A^* \}
\]

Hence \( \Lambda(\phi_{a^*}) \subseteq \Lambda(a) \) for every \( a^* \) in \( B^* \); so if \( \Delta \) denotes the closure of the union of the sets \( \Lambda(\phi_{a^*}) \) taken over all \( a^* \) in \( B^* \), then we have \( \Delta \subseteq \Lambda(a) \). If \( \Delta \) were a proper subset, then there would exist a point \( t_0 \) together with a closed neighborhood \( N \) of \( t_0 \) such that \( N \cap \Delta = \emptyset \). But then \( N \cap \Lambda(\phi_{a^*}) = \emptyset \) for every \( a^* \) in \( B^* \). Since \( \Delta(\phi_{a^*}) \) = hull \( I(\phi_{a^*}) \), we see that if \( \Lambda_f \subseteq N \) and \( f(t_0) \neq 0 \), then \( f \) belongs to \( I(\phi_{a^*}) \) for all \( a^* \) in \( B^* \). But then \( f \) belongs to \( I(a) \). Since the point \( t_0 \) belongs to \( \Lambda(a) \) we have \( f(t_0) = 0 \), which is a contradiction. Hence, using the fact that \( \Delta(\phi_{a^*}) = \sigma(\phi_{a^*}) \), we have \( \Lambda(a) = \Delta \) and \( \Delta = \text{closure of the union of the sets } \sigma(\phi_{a^*}) \) taken over all \( a^* \) in \( B^* \). On the other hand, it is easily seen that \( \Delta \subseteq \sigma(a) \). Just as before, if \( \Delta \) were a proper subset there would exist a point \( t_0 \) in \( \sigma(a) \) and a closed neighborhood \( N \) of \( t_0 \) such that \( N \cap \sigma(\phi_{a^*}) = \emptyset \) for all \( a^* \). But then \( a^* [(t - V)^{-1} a] \) would be analytic in \( N \) for every \( a^* \) in \( B^* \), contradicting the assumption that \( t_0 \) belongs to \( \sigma(a) \).

In the following generalized approximation theorem, the role played by the range of the spectral projections in the case of a unitary operator is taken over by certain “spectral subspaces” defined in the following way. For each closed subset \( \Delta \) of the circumference of the unit circle, set \( M_{\Delta} = \{ a \in B : \sigma(a) \subseteq \Delta \} \). Wermer [8] has shown that the linear manifold \( \{ a \in B : \Lambda_{a} \subseteq \Delta \} \) is closed; this, together with Theorem 1, shows that \( M_{\Delta} \) is indeed a subspace.

**Theorem 2.** Given any \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that for any \( \lambda \) with \( -\pi < \lambda < \pi \) and any element \( a \) in \( B \) which lies in the subspace \( L(\lambda) = \{ a \in B : \sigma(a) \subseteq [\lambda - \delta, \lambda + \delta] \} \)

\[
\| (V - e^{\lambda} a) \| \leq \epsilon \| a \|.
\]
Moreover, the space \( B \) is spanned by a finite collection of such manifolds.

The proof rests on the following lemma.

**Lemma 2.** For any \( \lambda \in [-\pi, \pi] \) and any \( \epsilon > 0 \) there exists an \( \eta > 0 \) and a function \( f \) in \( A_\alpha \) such that

(i) \( \|f\| = \sum_{n=0}^{\infty} \rho_n |f_n| < \epsilon \), and

(ii) \( f(\theta) = e^{i\lambda} - e^{i\eta} \) for \( |\theta - \lambda| < \eta \).

It suffices to find such an \( f \) in the case where \( \lambda = 0 \), since one has only to set \( f_n(\theta) = e^{i\lambda} f(\theta - \lambda) \), noting that \( \|f_\lambda\| = \|f\| \), to obtain the general result. For any \( \alpha > 0 \), set

\[
\begin{align*}
  f_\lambda(\theta) &= 1 - e^{i\theta} & \text{for} & \quad |\theta| \leq \alpha, \\
  &= 0 & \text{for} & \quad |\theta| \geq 2\alpha, \\
  \text{with} & \quad f \in C^\infty.
\end{align*}
\]

Then \( |f_\lambda| \leq K_\alpha \) and \( |f_n| \leq \alpha K_1 \ |n|^{-1} \) for any integer \( l \), where \( K_0 \) and \( K_1 \) depend on \( f \). Taking \( l = q + 2 \) and recalling that \( \rho_n \leq M \) \( 1 + |n|^{-q} \), we obtain \( \|f\| \leq \alpha K' \) for some constant \( K' \). Letting \( \eta = \epsilon / K' \) completes the proof of the lemma.

Now if \( \epsilon > 0 \) and \( \lambda \) are given, we construct \( f \) as in the above lemma and we set \( \delta = \eta / 2 \). Using Lemma 1(c) we see that if \( \sigma(a) \subseteq [\lambda - \delta, \lambda + \delta] \), then

\[
\begin{align*}
  f \circ a &= (e^{i\lambda} - e^{i\theta}) \circ a = (e^{i\lambda} - V)a.
\end{align*}
\]

Hence

\[
\| (e^{i\lambda} - V)a \| = \| f \circ a \| < \epsilon \|a\|.
\]

For the second part, we cover the interval \([-\pi, \pi]\) with any finite collection \( \{\Delta_j\}_{j=1}^p \) of overlapping intervals each having length equal to the \( \delta \) found in the first part. Over this cover we construct a partition of unity \( \{u_j\}_{j=1}^p \) with \( u_j \in C^\infty \). Then for any element \( a \) in \( B \) we have \( a = a_1 + a_2 + \cdots + a_n \), where \( a_j = u_j \circ a \), and \( \sigma(a_j) \subseteq \Delta_j \cap \sigma(a) \subseteq \Delta_j \). Let \( \lambda_j \) denote the center of \( \Delta_j \) and \( P_j(\theta) = e^{i\lambda_j} - e^{i\theta} \). Then, using the preceding lemma, there exists an \( f^{(j)} \) in \( A_\alpha \) such that \( \| f^{(j)} \| < \epsilon \) and \( f^{(j)} \equiv P_j \) in a neighborhood of \( \Delta_j \). Thus

\[
\| (e^{i\lambda_j} - V)a \| = \| f^{(j)} \circ a \| \leq \epsilon \|a\|.
\]

**Corollary.** If \( a \neq 0 \), then \( \sigma(a) = \{\lambda_0\} \) if and only if \( Va = e^{i\lambda_0}a \).

**References**

A NOTE ON NORMAL DILATIONS

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1. Introduction. Our purpose is to give certain sufficient conditions that a normal dilation of an operator be an extension from a reducing subspace. The first result of this kind, that we know of, is due to T. Andô [1] who considered compact normal dilations. In this note we use only assumptions about the nature of the spectrum; nevertheless, we are able to recover Andô's theorem.

Let $A$ be an operator on a Hilbert space $\mathcal{H}$. Let $P$ be the orthogonal projection of $\mathcal{H}$ onto a subspace $\mathcal{K}$. Let $T$ denote the restriction of $PAP$ to $\mathcal{K}$. The operator $T$ is called a compression of $A$ and $A$ is called a dilation of $T$. If $T^n$ is the compression of $A^n$ ($n = 0, 1, 2, 3, \ldots$) then $T$ is called a strong compression and $A$ a strong dilation. Let $X$ be a compact subset of the plane containing $\sigma(A)$ and $\sigma(T)$, the spectra of $A$ and $T$. The operator $A$ is said to be an $X$-dilation of $T$ if, for every rational function $r(\cdot)$ which is analytic on $X$, the operator $r(A)$ is a dilation of $r(T)$. These definitions were introduced by Halmos. Some other writers use "dilation" to mean what we call strong dilation. Sz-Nagy uses "projection" for compression. When $T$ is a strong compression of $A$ Andô calls $\mathcal{K}$ a "semi-invariant" subspace of $A$.

These notions are related to the more familiar concepts of invariant subspace and reducing subspace as follows. If $\mathcal{K}$ is an invariant subspace of $A$ then $A$ is a dilation of the restriction of $A$ to $\mathcal{K}$. Conversely, if $A$ is a dilation of a compression of a bounded operator, then $A$ is a strong compression of the original operator. In this note we shall use only these two results and abstract them to a more general setting.


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