AN APPROXIMATION THEOREM FOR A CLASS OF OPERATORS[1]

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It is well known that if U is a unitary operator in a Hilbert space H, then the following approximation theorem is an immediate consequence of the spectral representation for the operator U.

THEOREM A. (a) *Let* $\{E_t: 0 \leqq t \leqq 2\pi\}$ *denote the family of spectral projections associated with* U; *then if* $\epsilon > 0$ *and* α, $0 \leqq \alpha \leqq 2\pi$, *are given and if* x *is any element in the range of the projection* $E_{\alpha+\epsilon} - E_\alpha$, *we have*:

$$\|(U - e^{i\alpha})x\| \leqq \epsilon\|x\|;$$

(b) *moreover, for this same* ϵ, *there exists a finite collection of closed linear manifolds in* H *such that* H *is the direct sum of these manifolds*; *and in each subspace,* U *behaves as in part* (a).

E. R. Lorch [7] extended this theorem to certain classes of operators in a reflexive Banach space. The class considered by Lorch consists of those bounded, invertible operators V for which the norms of their iterates are restricted by the condition, $\|V^n\| = O(1)$ as $|n|$ tends to infinity.

The author [6] extended the results or Lorch to a somewhat larger class restricted by the condition, $\|V^n\| = O(|n|)$. This extension was made through the use of methods developed by N. Dunford [4], [5].

In the present paper, the result is extended to a much larger class of operators by using the methods of Harmonic analysis as developed by A. Beurling [1], J. Wermer [8], Y. Domar [3], and others. It should be mentioned that this extension might have been possible through the use of methods developed by F. Wolf [9] in his spectral theory for operators based on the generalized trigonometric integrals of S. Bochner. The present class is restricted by the condition,

(i) $\|V^n\| = O(|n|^q)$ as $|n|$ tends to infinity, for some $q > 0$.

An operator V, defined in a Banach space B, which satisfies condition (i) is easily seen to have it spectrum on the circumference of the unit circle. Moreover, the usual operational calculus may be extended by introducing a certain weighted algebra associated with the sequence $\{\|V^n\|: n = 0, \pm 1, \cdots\}$. Such algebras were introduced by Beurling [1], and later generalized by Wermer and Domar.

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DEFINITION 1. Let ρ_n and d_n, $n=0, \pm 1, \cdots$ be sequences of positive numbers such that $\rho_n \geqq 1$, $\rho_{n+m} \leqq \rho_n \rho_m$, and $\rho_n \leqq d_n$, where d_n is an even, increasing sequence. Denote by A_ρ the space of all continuous, periodic functions $f(\theta)$ whose Fourier coefficients f_n satisfy $\sum_{n=-\infty}^{\infty} \rho_n |f_n| < \infty$.

When given the norm $\|f\| = \sum_{-\infty}^{\infty} \rho_n |f_n|$, the space A_ρ becomes a commutative Banach algebra under pointwise multiplication More-over, if ρ_n is further restricted by $\sum_{-\infty}^{\infty} \log \rho_n/(1+n^2) < \infty$, the algebra A_ρ is regular. In the present application, we take $\rho_n = \| V^n \|$; the relevant conditions satisfied by ρ_n follow directly from the restriction of polynomial growth on $\| V^n \|$.

Using the algebra A_ρ, an operational calculus may be generated for V by setting $f(V) = \sum_{-\infty}^{\infty} f_n V^n$, for such f in A_ρ. In order to apply the methods of harmonic analysis, we shall drop down to the space B and look at the induced map on $A_\rho \times B \to B$ defined by $f \circ a = f(V)a$, where f belongs to A_ρ and a to B. The idea of considering the induced map is an adaptation of a technique developed by Domar in his study of function algebras over locally compact Abelian groups.

Using the continuity of the mapping $(f, a) \to f \circ a$, we may associate with each element a in B a closed subset $\Lambda(a)$ of the circumference of the unit circle defined as the hull of the ideal $I(a) = \{f \in A_\rho : f \circ a = 0\}$. In addition, for each f in A_ρ, let Λ_f denote the closure of the support of f; then the following results are basic (cf. [3]).

LEMMA 1. (a) $\Lambda(a)$ *is void if and only if* $a=0$.

(b) *For any* f *in* A_ρ *and* a *in* B, $\Lambda(f \circ a) \subseteq \Lambda_f \cap \Lambda(a)$.

(c) *If* f *and* g *belong to* A_ρ *and* $f \equiv g$ *in some neighborhood of* $\Lambda(a)$, *then* $f \circ a = g \circ a$.

The elements ϕ of the adjoint space A_ρ^* may be identified with the space of sequences $\{\phi_n\}$ for which $\sup_n |\phi_n|/\rho_n < \infty$ by means of the representation $\phi(f) = \sum_{-\infty}^{\infty} f_n \phi_n$; furthermore $\|\phi\| = \sup_n |\phi_n|/\rho_n$. With each ϕ in A_ρ^*, there is associated a pair of functions $\Phi^+(z) = \sum_1^\infty \phi z^{-n}$, $\Phi^-(z) = -\sum_0^\infty \phi_{-n} z^n$, where Φ^+ and Φ^- are defined and analytic for $|z| > 1$ and $|z| < 1$, respectively. Using the pair (Φ^+, Φ^-) we may define a closed set $\sigma(\phi)$ as the set of points λ on the circumference of the unit circle for which the pair (Φ^+, Φ^-) do not continue each other across any arc containing the point λ (cf. [2], [8]).

On the other hand we may construct another representation of A_ρ, this time as an operator over the Banach space A_ρ^* in place of the space B. In this case the action of f as an operator in A_ρ^* will be defined by $(f \circ \phi)(g) = \phi(fg)$. With this representation the set $\Lambda(\phi)$ may be defined just as was done for the elements of B. In [3] it was shown that $\sigma(\phi)=\Lambda(\phi)$. Now for any a in B and a^* in B^*, the sequence $\phi_n=a^*(V^{n-1}a)$ defines an element ϕ in A_ρ^*. Using the definition of $\sigma(\phi)$ we are led to the usual definition (cf. [5]) of the spectrum $\sigma(a)$ of an element a in B as the set of points λ, $|\lambda|=1$, such that for some a^* in B^*, λ belongs to the spectrum of $\{a^*(V^{n-1}a)\}$ as an element of A_ρ^*.

THEOREM 1. *For each a in B, $\sigma(a)=\Lambda(a)$.*

This result is established as follows. Let a be a fixed element in B, and let ϕ_{a^*} denote the element in A_ρ^* corresponding to the sequence $\{a^*(V^na)\}$. Observe that since $(fg)\circ a=g\circ(f\circ a)$, we have

$$\begin{aligned}\bigcap\{I(\phi_{a^*})\colon a^* \text{ in } A_\rho^*\}\\ &= \{f \text{ in } A_\rho\colon a^*[g\circ(f\circ a)] = 0 \text{ for all } g \text{ in } A_\rho \text{ and } a^* \text{ in } A_\rho^*\}\\ &= \{f \text{ in } A_\rho\colon g\circ(f\circ a) = 0 \text{ for all } g \text{ in } A_\rho\} = I(a).\end{aligned}$$

Hence $\Lambda(\phi_{a^*})\subseteq\Lambda(a)$ for every a^* in B^*; so that if Δ denotes the closure of the union of the sets $\Lambda(\phi_{a^*})$ taken over all a^* in B^*, then we have $\Delta\subseteq\Lambda(a)$. If Δ were a proper subset, then there would exist a point t_0 together with a closed neighborhood N of t_0 such that $N\cap\Delta=\emptyset$. But then $N\cap\Lambda(\phi_{a^*})=\emptyset$ for every a^* in B^*. Since $\Lambda(\phi_{a^*})=\text{hull } I(\phi_{a^*})$, we see that if $\Lambda_f\subset N$ and $f(t_0)\neq 0$, then f belongs to $I(\phi_{a^*})$ for all a^* in B^*. But then f belongs to $I(a)$. Since the point t_0 belongs to $\Lambda(a)$ we have $f(t_0)=0$, which is a contradiction. Hence, using the fact that $\Lambda(\phi_{a^*})=\sigma(\phi_{a^*})$, we have $\Lambda(a)=\Delta$ and $\Delta=$closure of the union of the sets $\sigma(\phi_{a^*})$ taken over all a^* in B^*. On the other hand, it is easily seen that $\Delta\subseteq\sigma(a)$. Just as before, if Δ were a proper subset there would exist a point t_0 in $\sigma(a)$ and a closed neighborhood N of t_0 such that $N\cap\sigma(\phi_{a^*})=\emptyset$ for all a^*. But then $a^*[(t-V)^{-1}a]$ would be analytic in N for every a^* in B^*, contradicting the assumption that t_0 belongs to $\sigma(a)$.

In the following generalized approximation theorem, the role played by the range of the spectral projections in the case of a unitary operator is taken over by certain "spectral subspaces" defined in the following way. For each closed subset Δ of the circumference of the unit circle, set $M_\Delta=\{a \text{ in } B\colon\sigma(a)\subseteq\Delta\}$. Wermer [8] has shown that the linear manifold $\{a \text{ in } B\colon\Lambda_a\subseteq\Delta\}$ is closed; this, together with Theorem 1, shows that M_Δ is indeed a subspace.

THEOREM 2. *Given any $\epsilon>0$ there exists a $\delta>0$ such that for any λ with $-\pi<\lambda<\pi$ and any element a in B which lies in the subpace $L(\lambda)=\{a \text{ in } B;\ \sigma(a)\subseteq[\lambda-\delta,\lambda+\delta]\}$*

$$\|(V-e^{i\lambda})a\|\leqq\epsilon\|a\|.$$

Moreover, the space B is spanned by a finite collection of such manifolds.

The proof rests on the following lemma.

LEMMA 2. *For any λ in $[-\pi, \pi]$ and any $\epsilon > 0$ there exists an $\eta > 0$ and a function f in A_ρ such that*
(i) $\|f\| = \sum_{-\infty}^{\infty} \rho_n |f_n| < \epsilon$, *and*
(ii) $f(\theta) \equiv e^{i\lambda} - e^{i\theta}$ *for* $|\theta - \lambda| < \eta$.

It suffices to find such an f in the case where $\lambda = 0$, since one has only to set $f_\lambda(\theta) = e^{i\lambda} f(\theta - \lambda)$, noting that $\|f_\lambda\| = \|f\|$, to obtain the general result. For any $\alpha > 0$, set

$$f(\theta) = 1 - e^{i\theta} \qquad \text{for } |\theta| \leqq \alpha,$$
$$= 0 \qquad \text{for } |\theta| \geqq 2\alpha, \text{ with } f \text{ in } C^\infty.$$

Then $|f_0| \leqq K_0 \alpha$ and $|f_n| \leqq \alpha K_l |n|^{-l}$ for any integer l, where K_0 and K_l depend on f. Taking $l \geqq q+2$ and recalling that $\rho_n \leqq M(1+|n|^q)$, we obtain $\|f\| \leqq \alpha K'$ for some constant K'. Letting $\eta = \epsilon / K'$ completes the proof of the lemma.

Now if $\epsilon > 0$ and λ are given, we construct f as in the above lemma and we set $\delta = \eta/2$. Using Lemma 1(c) we see that if $\sigma(a) \subseteq [\lambda - \delta, \lambda + \delta]$, then

$$f \circ a = (e^{i\lambda} - e^{i\theta}) \circ a = (e^{i\lambda} - V)a.$$

Hence

$$\|(e^{i\lambda} - V)a\| = \|f \circ a\| < \epsilon \|a\|.$$

For the second part, we cover the interval $[-\pi, \pi]$ with any finite collection $\{\Delta_j\}_{j=1}^n$ of overlapping intervals each having length equal to the δ found in the first part. Over this cover we construct a partition of unity $\{u_j\}_{j=1}^n$ with $u_j \in C^\infty$. Then for any element a in B we have $a = a_1 + a_2 + \cdots + a_n$, where $a_j = u_j \circ a$, and $\sigma(a_j) \subseteq \Lambda(u_j) \cap \sigma(a) \subseteq \Delta_j$. Let λ_j denote the center of Δ_j and $P_j(\theta) = e^{i\lambda_j} - e^{i\theta}$. Then, using the preceding lemma, there exists an $f^{(j)}$ in A_ρ such that $\|f^{(j)}\| < \epsilon$ and $f^{(j)} \equiv P_j$ in a neighborhood of Δ_j. Thus

$$\|(e^{i\lambda_j} - V)a\| = \|f^{(j)} \circ a_j\| \leqq \epsilon \|a_j\|.$$

COROLLARY. *If $a \neq 0$, then $\sigma(a) = \{\lambda_0\}$ if and only if $Va = e^{i\lambda_0} a$.*

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A NOTE ON NORMAL DILATIONS

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I. **Introduction.** Our purpose is to give certain sufficient conditions that a normal dilation of an operator be an extension from a reducing subspace. The first result of this kind, that we know of, is due to T. Andô [1] who considered compact normal dilations. In this note we use only assumptions about the nature of the spectrum; nevertheless, we are able to recover Andô's theorem.

Let A be an operator on a Hilbert space $\mathcal{K}$. Let P be the orthogonal projection of $\mathcal{K}$ onto a subspace $\mathcal{H}$. Let T denote the restriction of PAP to $\mathcal{H}$. The operator T is called a *compression* of A and A is called a *dilation* of T. If T^n is the compression of A^n $(n=0\ 1, 2, 3, \cdots)$ then T is called a *strong compression* and A a *strong dilation*. Let X be a compact subset of the plane containing $\sigma(A)$ and $\sigma(T)$, the spectra of A and T. The operator A is said to be an *X-dilation* of T if, for every rational function $r(\cdot)$ which is analytic on X, the operator $r(A)$ is a dilation of $r(T)$. These definitions were introduced by Halmos. Some other writers use "dilation" to mean what we call strong dilation. Sz-Nagy uses "projection" for compression. When T is a strong compression of A Andô calls $\mathcal{H}$ a "semi-invariant" subspace of A.

These notions are related to the more familiar concepts of invariant subspace and reducing subspace as follows. If $\mathcal{H}$ is an invariant sub-

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