AN INVARIANCE THEOREM FOR REPRESENTATIONS OF BANACH ALGEBRAS

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1. **Introduction.** Under certain circumstances, a topological linear space acted upon by a given linear transformation τ can be shown to admit the action of a suitably continuous one-parameter group of transformations containing τ, and associated with τ in a natural way. An illustration is afforded by the following theorem, which plays an important auxiliary role in [1]:

Let $\mathcal{S}$ be a norm-closed linear subspace of $\mathfrak{L}(\mathfrak{H})$. Then if $\mathcal{S}$ is invariant under $A \rightarrow T^{-1}AT$, $\mathcal{S}$ is invariant also under $A \rightarrow A \log T - (\log T)A$, and under $A \rightarrow T^{-s}AT^{s}$ for all real numbers s.

(Here $\mathfrak{H}$ is a Hilbert space, and $\mathfrak{L}(\mathfrak{H})$ the algebra of all bounded linear operators on $\mathfrak{H}$. T is a positive, regular element of $\mathfrak{L}(\mathfrak{H})$.)

It is the purpose of this note to provide an appropriately general setting for this theorem. Our main result is the:

INVARIANCE THEOREM. *Let $\mathfrak{B}$ be a Banach algebra, x and y elements of $\mathfrak{B}$ with spectra in the open right half-plane H^+. Let ϕ be a strongly continuous representation of $\mathfrak{B}$ on the topological linear space $\mathcal{E}$, with $\phi(xy) = \phi(yx)$. Then every closed subspace $\mathcal{S}$ of $\mathcal{E}$ invariant under $\phi(xy)$ is invariant also under $\phi(\log x + \log y)$ and under $\phi(x^s y^s)$ for all real numbers s. $s \rightarrow \phi(x^s y^s)$ is a strongly continuous one-parameter group in $\mathfrak{L}(\mathcal{S})$.*

(Here $z^s = \exp(s \log z)$, where log is defined for complex α in the complement Σ of the nonpositive real numbers R^- by taking arg $\alpha \in (-\pi, \pi)$, and then for z in $\mathfrak{B}$ with spectrum $\sigma(z) \subset \Sigma$ by Cauchy's formula [2, §§5.2, 5.4], [5, Theorem 3.5.1].)

2. **Preliminary results.**

LEMMA 1. *Let $\mathfrak{B}$ be a Banach algebra with identity element, x and y commuting regular elements of $\mathfrak{B}$ with spectra in H^+. Then* $\log(xy) = \log x + \log y$.

PROOF. We first remark that $\sigma(xy) \subset \sigma(x)\sigma(y) = \{rs \mid r \in \sigma(x), s \in \sigma(y)\}$. In fact, let $\mathfrak{B}'$ be a maximal commutative subalgebra of $\mathfrak{B}$

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containing x and y. Then the injection map of $\mathfrak{B}'$ into $\mathfrak{B}$ is spectrum-preserving [5, Theorem 1.6.9]. If $\alpha \in \sigma(xy)$, we can choose a homomorphism h of $\mathfrak{B}'$ onto the complex numbers with $\alpha = h(xy) = h(x)h(y) \in \sigma(x)\sigma(y)$, proving the remark. (The author is indebted to Professor R. V. Kadison for the simplicity of the above proof.)

From the hypothesis and the remark above, we conclude that $\sigma(xy) \subset \Sigma$, so that $\log(xy)$ is defined. By standard properties of the functional calculus for commutative Banach algebras [5, Theorem 3.5.1] we have for each element z of $\mathfrak{B}'$ with $\sigma(z) \subset \Sigma$, and for each homomorphism h of $\mathfrak{B}'$ onto the complex numbers,

$$h(\log z) = \log h(z).$$

Now $\log(h(x)h(y)) = \log h(x) + \log h(y)$, since $|\arg h(x)| < \pi/2$, $|\arg h(y)| < \pi/2$, showing the imaginary part of the sum on the right to be less than π in absolute value.

We set $q = \log(xy) - \log x - \log y$ and show by a trivial computation that $h(q) = 0$, whence $q \in \text{radical } \mathfrak{B}'$. On the other hand, $\exp(q) = 1$, so that by a result of Lorch [3, p. 421; 2, Theorem 5.5.5], q is a finite linear combination of idempotents j, which we may take to be mutually orthogonal. But then, since the radical is an ideal, each j is in the radical. Finally, an idempotent in the radical must be zero, and $q = 0$, as claimed.

Mergelyan [4] has shown that if E is a compact subset of the complex plane, each function f continuous on E and analytic at interior points of E is uniformly approximable on E by polynomial functions if and only if E does not divide the plane. For z in $\mathfrak{B}$ with $\sigma(z) \subset \Sigma$, we apply Mergelyan's theorem to the case $f = \log$, $E = E' + V$, where $E' = \{\alpha \mid |\alpha| \leqq \|z\| \ \& \ \text{dist}(\alpha, R^-) \geqq 2/3 \ \text{dist}(\sigma(z), R^-)\}$ and V is the closed disc of radius $1/2 \ \text{dist}(\sigma(z), R^-)$ centered at 0 in the complex plane. Then if $\{p_n\}$ is a sequence of polynomials such that $p_n \to \log$ uniformly on E, $p_n(z) \to \log(z)$ by [2, Theorem 5.2.5]. Thus $\log z$, $s \log z$, and finally $\exp(s \log z) = z^s$ are approximable by polynomials in z, provided only that $\sigma(z) \subset \Sigma$. An immediate consequence is that if w and z are commuting elements of $\mathfrak{B}$ with spectra in Σ, then $\log w$, $\log z$, w^s and z^s lie in the closed commutative algebra generated by w, z and the unit element e. Then, returning to the notation and hypotheses of Lemma 1 we have:

COROLLARY 1. $x^s y^s = (xy)^s$.

PROOF. Clear.

COROLLARY 2. *$x^s y^s$ is a limit of polynomials in xy.*

This is immediate from Corollary 1 and the preceding remarks.

3. **Proof of the main result.** The strong continuity of ϕ means that for each vector v of $\mathcal{E}$, the mapping $x \to \phi(x)v$ is continuous on $\mathfrak{B}$ to $\mathcal{E}$. Equivalently, ϕ is continuous on $\mathfrak{B}$ with its norm topology to $\mathfrak{L}(\mathcal{E})$ in the point-open topology. Thus the kernel $\mathfrak{M}$ of ϕ is a closed, two-sided ideal in $\mathfrak{B}$, and ϕ defines a strongly continuous and faithful representation ϕ_* of $\mathfrak{B}/\mathfrak{M}$ on $\mathcal{E}$. Our proof is a straightforward reduction from $\mathfrak{B}$ to $\mathfrak{B}/\mathfrak{M}$.

The natural map μ of $\mathfrak{B}$ onto $\mathfrak{B}/\mathfrak{M}$ is continuous, and preserves polynomials, so that $\mu \circ \exp = \exp \circ \mu$. Moreover, since $\sigma(z_*) \subset \sigma(z)$ (for z in $\mathfrak{B}$, $z_* = \mu(z)$), we may conclude that if $\sigma(z) \subset H^+$(resp. Σ), then $\sigma(z_*) \subset H^+$ (resp. Σ), and in either case $\log z_*$ exists and $(\log z)_* = \log(z_*)$, $(z^s)_* = (z_*)^s$. Finally, $x_* y_* = y_* x_*$, because $\phi(xy) = \phi(yx)$ by hypothesis. Applying Corollary 2 (and its proof) to these commuting elements of $\mathfrak{B}/\mathfrak{M}$, we conclude that $x_*^s y_*^s$ and $\log(x_*^s y_*^s)$ are limits of polynomials in $x_* y_*$. But then if $\mathcal{S}$ is a closed subspace of $\mathcal{E}$ invariant under $\phi(xy) = \phi_*(x_* y_*)$, $\mathcal{S}$ is invariant also under $\phi(x_*^s y_*^s) = \phi_*(x_*^s y_*^s)$ and under $\phi(\log x + \log y) = \phi_*(\log x_* + \log y_*) = \phi_*(\log(x_* y_*))$, which was to be shown.

That $s \to \phi(x^s y^s)$ is a strongly continuous group of endomorphisms of $\mathcal{S}$ is clear from the above considerations.

4. **Application to inner automorphisms.** In this section, we recover the first-cited theorem of the introduction, in considerably greater generality.

COROLLARY 3. *Let $\mathfrak{A}$ be a Banach algebra with unit element, z an element of $\mathfrak{A}$ with $\sigma(z) \subset H^+$. Let $\mathcal{S}$ be a closed subspace of $\mathfrak{A}$ invariant under $x \to z^{-1}xz$. Then $\mathcal{S}$ is invariant under $x \to x \log z - (\log z)x$ and under $x \to z^{-s}xz^s$ for all real numbers s.*

PROOF. We know that z^{-1} exists, and that $\sigma(z^{-1}) \subset H^+$, also. In the statement of the theorem, take $\mathfrak{B} = \mathfrak{L}(\mathfrak{A})$, $\mathcal{E} = \mathfrak{A}$, and $\phi = $ identity. Denote by $w \otimes I$ ($I \otimes w$) the transformation $v \to wv$ ($v \to vw$) for w in $\mathfrak{A}$, and set $x = z^{-1} \otimes I$, $y = I \otimes z$. Recalling that the left- and right-regular representations of a Banach algebra $\mathfrak{A}$ with identity are spectrum-preserving isometric isomorphisms of $\mathfrak{A}$ onto mutually commuting subalgebras of $\mathfrak{L}(\mathfrak{A})$, [**5**, Chapter 1] we find the hypotheses of the theorem verified in the above interpretation. We conclude from the theorem that $\mathcal{S}$ is invariant under $\log(z^{-1} \otimes I) + \log(I \otimes z)$ and under $(z^{-1} \otimes I)^s (I \otimes z)^s$ for all real s. But these are $I \otimes \log z - \log z \otimes I$ and $z^{-s} \otimes z^s$, respectively. In fact, polynomial approximations via Mergelyan's theorem, together with the remark above on the regular representations, afford immediate verification.

BIBLIOGRAPHY

1. L. T Gardner, *On isomorphisms of C*-algebras*, Amer. J. Math. **87** (1965), 384–396.

2. E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloq. Publ. Vol. 30, Amer. Math. Soc., Providence, R. I., 1957.

3. E. R. Lorch, *The theory of analytic functions in normed abelian vector rings*, Trans. Amer. Math. Soc. **54** (1943), 414–425.

4. S. N. Mergelyan, *On the representation of functions by series of polynomials*, Dokl. Akad. Nauk SSSR **78** (1951), 405–408; English Transl., Amer. Math. Soc. Transl. No. 85, Amer. Math. Soc., Providence, R. I., 1953.

5. C. E. Rickart, *General theory of Banach algebras*, Van Nostrand, New York, 1960.

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