HOMOLOGY OF DELETED PRODUCTS IN DIMENSION ONE

ARTHUR H. COPELAND, JR.[1]

If X is a topological space, then the *deleted product space* of X is the difference $RX = X \times X - \Delta X$, where ΔX is the diagonal subspace of the topological product $X \times X$. An embedding $f: X \to Y$ induces a map $Rf: RX \to RY$, and isotopic embeddings induce homotopic (in fact, isotopic) maps. Thus $X \to H_k(RX)$ is an isotopy functor when $H_k(\ )$ is the kth singular homology functor with integral coefficients [1]. The first section of this note shows that these isotopy functors pick out the isotopy equivalences between 1-dimensional finitely triangulable spaces. The second section presents a formula for computing $H_k(RX)$ when X is a 1-dimensional finitely triangulable space.

1. Isotopy equivalences.

THEOREM. *If X and Y are finitely triangulable 1-dimensional spaces and if $f: X \to Y$ is an embedding, then*

$$Rf_*: H_k(RX) \to H_k(RY)$$

is an isomorphism for all k if and only if f is an isotopy equivalence.

PROOF. S.-T. Hu has shown that if f is an isotopy equivalence, then Rf_* is an isomorphism in all dimensions [1].

Suppose Rf_* is an isomorphism. If X has components $X_1, \cdots, X_n$, then

$$RX = \bigcup_{i=1}^{n} RX_i \cup \bigcup_{i \neq j} X_i \times X_j. \tag{1}$$

The summands in this union are mutually disjoint and are closed in RX. Thus if f induces an isomorphism of the 0-dimensional homology groups of the deleted products, then f must induce a one-to-one correspondence between the components of X and those of Y. It follows that f induces isomorphisms between the homology groups of the deleted products of corresponding components. It suffices, then, to prove the theorem when X and Y are both connected.

The embedding $f: X \to Y$ may be regarded as an inclusion map, and the two spaces may be given triangulations which are consistent with

Received by the editors September 21, 1964.

[1] This research was supported by NSF Grant G24154.

each other. Next, factor the inclusion map $f: X = X_0 \to X_1 \to \cdots \to X_m = Y$ in such a way that X_i is formed from X_{i-1} by the addition of a single 1-simplex σ. Take the triangulations sufficiently fine that these additions are of one of the following two types.

(a) Addition at both ends: $\sigma \cap X_{i-1}$ consists of the two endpoints of σ, and each has order 2 in X_i.

(b) Addition at one end: $\sigma \cap X_{i-1}$ is one of the two endpoints of σ and is a vertex of order s in X_i.

The *order* of a vertex is the number of 1-simplexes which meet it. C. W. Patty [2] shows that with an addition at both ends, the inclusion map induces a homomorphism $H_1(RX_{i-1}) \to H_1(RX_i)$ which is never an epimorphism, and that with an addition at one end, the induced homomorphism is an epimorphism only when the vertex has order $s=2$ in X_i. Thus if Rf_* is an isomorphism, each inclusion must be of the second type and have $s=2$. But such an embedding is an isotopy equivalence, whence f is, also.

Note that the proof of necessity uses only the fact that Rf_* is an epimorphism in dimensions $k=0, 1$.

It is not true that X and Y must have the same isotopy type whenever RX and RY have isomorphic homology. The next section provides a wealth of counterexamples.

2. **Betti numbers of deleted products.** It is an immediate corollary to Patty's work that the homology groups $H_k(RX)$ are free abelian when X is a 1-dimensional finitely triangulable space. Thus in order to describe these groups, it suffices to describe the k-dimensional Betti numbers β_k of RX for each k. Note that these are zero for $k>2$.

The 0*-dimensional Betti number* β_0. If X consists of a single point, then RX is empty. If X is an arc, then RX has two components. If X is any other connected space, then RX is connected. Thus if X has n components, if p of these are isolated points and if q are arcs, then it follows from formula (1) of §1 that

$$\beta_0 = n^2 - p + q.$$

The 2*-dimensional Betti number* β_2. The group $H_2(X \times X)$ is generated by the 2-dimensional cycles of the tori $c \times c' \subset X \times X$, where c and c' are simple closed curves in X. The inclusion map induces a monomorphism $H_2(RX) \to H_2(X \times X)$, and the image is generated by the cycles $c \times c'$ with c and c' disjoint simple closed curves. The rank of this image is β_2.

The 1*-dimensional Betti number* β_1. Let X be given a fixed triangulation. W.-T. Wu [3] has shown that the inclusion map of

$$JX = \bigcup \{x \times y : x, y \text{ simplexes of } X \text{ and } x \cap y \text{ empty}\}$$

in RX is a homotopy equivalence. The cells $x \times y$ form a cellular decomposition of JX. For $k=0, 1, 2$, let c_k and c_k' be the number of k-dimensional cells in X and JX, respectively. If x is a simplex of X, let $a(x)$ be the number of 1-simplexes in X which meet x; x itself is numbered among these in case it is a 1-simplex. If u_i $(i=1, \cdots, c_0)$ are the 0-simplexes of X and if v_i $(i=1, \cdots, c_1)$ are the 1-simplexes, then

$$c_0' = c_0^2 - c_0,$$

$$c_1' = 2 \sum_{i=1}^{c_0} (c_1 - a(u_i)) = 2c_0c_1 - 4c_1,$$

$$c_2' = \sum_{i=1}^{c_1} (c_1 - a(v_i)) = c_1^2 - \sum_{i=1}^{c_1} a(v_i).$$

It follows at once from these three equations and the Euler-Poincaré formula that

$$\beta_1 = \beta_0 + \beta_2 - 4c_1 + c_0 - (c_0 - c_1)^2 + \sum_{i=1}^{c_1} a(v_i).$$

References

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3. W.-T. Wu, *On the realization of complexes in euclidean spaces*. I, Acta Math. Sinica **5** (1955), 505–552.

Northwestern University