A NOTE ON THE PARALLELIZABILITY OF REAL STIEFEL MANIFOLDS

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1. **Introduction.** Sutherland has proved in [4], among other things, that the Stiefel manifolds $V_{n,q}$ of orthonormal q-frames in R^n are parallelizable for $q \geqq 2$. The proof there consists of showing that the $V_{n,q}$ are stably parallelizable, and then invoking some results of Adams and Kervaire, [1], [2], and [3], which show that under certain circumstances *stably parallelizable* implies *parallelizable.* The purpose of this note is to give a more elementary proof of the parallelizability of the $V_{n,q}$ for $q > 2$.

2. **Preliminaries and statement of the theorem.** Let S^{n-1} denote the unit sphere in R^n. $V_{n,q}$ can then be regarded as the subspace of $S^{n-1}\times \cdots \times S^{n-1}$, q times, consisting of all $(x_1, \cdots, x_q)$ in $S^{n-1} \times \cdots \times S^{n-1}$ with $x_i \perp x_j$ for $i \neq j$. As such, $V_{n,q}$ is a differentiable submanifold of $S^{n-1}\times \cdots \times S^{n-1}$.

THEOREM. *$V_{n,q}$ has a trivial normal bundle in $S^{n-1}\times \cdots \times S^{n-1}$. If $q > 2$, then $V_{n,q}$ is parallelizable.*

For any bundle ξ, write ξ_x for the fibre over a point x in the base space. Write $\bar{\tau}$ for the tangent bundle of $S^{n-1}\times \cdots \times S^{n-1}$, τ for the tangent bundle of $V_{n,q}$ and ν for the normal bundle of $V_{n,q}$ in $S^{n-1} \times \cdots \times S^{n-1}$. Then $\bar{\tau} | V_{n,q} = \tau \oplus \nu$, where $\oplus$ denotes Whitney sum.

If $x = (x_1, \cdots, x_q) \in S^{n-1} \times \cdots \times S^{n-1}$, $\bar{\tau}_x$ consists of all $(x; u_1, \cdots, u_q)$ where $u_i \in R^n$, $u_i \perp x_i$, $1 \leqq i \leqq q$. The inner product of $(x; u_1, \cdots, u_q)$ and $(x; u_1', \cdots, u_q')$ in $\bar{\tau}_x$ is $\sum_i u_i \cdot u_i'$. For $x \in V_{n,q}$, τ_x is the orthogonal complement of ν_x in $(\bar{\tau} | V_{n,q})_x$.

3. **The normal bundle ν.** If $x \in V_{n,q}$, then $(x; u_1, \cdots, u_q) \in (\bar{\tau} | V_{n,q})_x$ is normal to $V_{n,q}$ if and only if

$$\lim_{x' \to x; x' \in V_{n,q}} \frac{(x - x') \cdot (u_1, \cdots, u_q)}{\|x - x'\|} = 0. \tag{1}$$

For $1 \leqq r < s \leqq q$, let $\lambda_{r,s}$ denote the line subbundle of $\bar{\tau} | V_{n,q}$ whose fibre $(\lambda_{r,s})_x$ is spanned by

Received by the editors September 6, 1964.

[1] The author held a National Science Foundation Summer Fellowship for Graduate Teaching Assistants during the preparation of this note.

$$(x; 0, \cdots, 0, \underset{r\text{th}}{x_s}, 0, \cdots, 0, \underset{s\text{th}}{x_r}, 0, \cdots, 0).$$

$\lambda_{r,s}$ is a trivial bundle, and for $(r, s) \neq (r', s')$, $(\lambda_{r,s})_x$ is orthogonal to $(\lambda_{r',s'})_x$ for all $x \in V_{n,q}$. Thus, $\oplus_{1 \leqq r < s \leqq q} \lambda_{r,s}$ is a trivial $\frac{1}{2}q(q-1)$-plane subbundle of $\bar{\tau} | V_{n,q}$.

If $x = (x_1, \cdots, x_q)$ and $x' = (x_1', \cdots, x_q')$ are in $V_{n,q}$, then using the fact that $x_i \cdot x_j = x_i' \cdot x_j' = \delta_{ij}$, we have

$$\left| \frac{(x - x') \cdot (0, \cdots, 0, \underset{r\text{th}}{x_s}, 0, \cdots, 0, \underset{s\text{th}}{x_r}, 0, \cdots, 0)}{\| x - x' \|} \right|$$

$$= \frac{| (x_r - x_r') \cdot x_s + (x_s - x_s') \cdot x_r |}{(\sum_i \| x_i - x_i' \|^2)^{1/2}} = \frac{| (x_r - x_r') \cdot (x_s - x_s') |}{(\sum_i \| x_i - x_i' \|^2)^{1/2}}$$

$$\leqq \| x_r - x_r' \| \cdot \frac{\| x_s - x_s' \|}{(\sum_i \| x_i - x_i' \|^2)^{1/2}} \leqq \| x_r - x_r' \| .$$

As x' tends to x, x_r' tends to x_r, so $\| x_r - x_r' \|$ tends to 0. Hence by (1), $(\lambda_{r,s})_x$ is normal to $V_{n,q}$ at x for all $x \in V_{n,q}$, $1 \leqq r < s \leqq q$ and so $\oplus_{1 \leqq r < s \leqq q} \lambda_{r,s}$ is a trivial $\frac{1}{2}q(q-1)$-plane subbundle of ν.

But since $V_{n,q}$ is an S^{n-q}-bundle over $V_{n,q-1}$, and $V_{n,1} \cong S^{n-1}$, it follows by induction on q that $V_{n,q}$ is a manifold of dimension $q(n-1) - \frac{1}{2}q(q-1)$. Hence, since dim $S^{n-1} \times \cdots \times S^{n-1}$, q times, is $q(n-1)$, it follows that ν has fibre dimension $\frac{1}{2}q(q-1)$. Hence, $\nu = \oplus_{1 \leqq r < s \leqq q} \lambda_{r,s} \cong \frac{1}{2}q(q-1)$ (we write k for the trivial k-plane bundle over $V_{n,q}$).

4. The tangent bundle τ. For $1 \leqq i \leqq q$, let α_i denote the subbundle of $\bar{\tau} | V_{n,q}$ whose fibre $(\alpha_i)_x$ consists of all

$$(x; 0, \cdots, 0, \underset{i\text{th}}{u}, 0, \cdots, 0)$$

where $u \in R^n$, $u \perp x_k$, $1 \leqq k \leqq q$. α_i is an $(n-q)$-plane bundle. Note that $(\alpha_i)_x$ is orthogonal to ν_x in $(\bar{\tau} | V_{n,q})_x$, $1 \leqq i \leqq q$.

For $1 \leqq r < s \leqq q$, let $\beta_{r,s}$ denote the line subbundle of $\bar{\tau} | V_{n,q}$ whose fibre $(\beta_{r,s})_x$ is spanned by

$$(x; 0, \cdots, 0, \underset{r\text{th}}{x_s}, 0, \cdots, 0, \underset{s\text{th}}{-x_r}, 0, \cdots, 0).$$

$\beta_{r,s}$ is a trivial line subbundle of $\bar{\tau} | V_{n,q}$. Note that $(\beta_{r,s})_x$ is orthogonal to ν_x, to all $(\alpha_i)_x$, and to $(\beta_{r',s'})_x$ for $(r, s) \neq (r', s')$. Hence $(\oplus_{1 \leqq i \leqq q} \alpha_i)$

$\oplus(\oplus_{1\leqq r<s\leqq q}\ \beta_{r,s})$ is asubbundle of τ of fibre dimension $q(n-q)+\frac{1}{2}q(q-1)=q(n-1)-\frac{1}{2}q(q-1)=\dim\ V_{n,q}$. Hence,

$$\tau=\left(\bigoplus_{1\leqq i\leqq q}\alpha_i\right)\oplus\left(\bigoplus_{1\leqq r\leqq s\leqq q}\beta_{r,s}\right)\cong\left(\bigoplus_i\alpha_i\right)\oplus\frac{1}{2}q(q-1).$$

The trivial n-plane bundle $V_{n,q}\times R^n$ over $V_{n,q}$ splits as the Whitney sum $\alpha\oplus\gamma$ where α_x consists of all (x, u), $u\in R^n$, $u\perp x_k$, $1\leqq k\leqq q$, and γ_x consists of all (x, v), $v\in R^n$, v in the span of $x_1, \cdots, x_q$. γ is a trivial q-plane bundle, having the q everywhere linearly-independent cross-sections s_i defined by $s_i(x)=(x, x_i)$, $1\leqq i\leqq q$. Hence, $\alpha\oplus q\simeq n$. Hence, $k\alpha\oplus l$ is trivial if $l\geqq q$, and k is any positive integer.

Each of the α_i above is equivalent to α. Hence $\tau\cong q\alpha\oplus\frac{1}{2}q(q-1)$. If $q>2$, $\frac{1}{2}q(q-1)\geqq q$, and so τ is trivial.

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