A NOTE ON THE PARALLELIZABILITY OF REAL STIEFEL MANIFOLDS

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1. Introduction. Sutherland has proved in [4], among other things, that the Stiefel manifolds $V_{n,q}$ of orthonormal $q$-frames in $\mathbb{R}^n$ are parallelizable for $q \geq 2$. The proof there consists of showing that the $V_{n,q}$ are stably parallelizable, and then invoking some results of Adams and Kervaire, [1], [2], and [3], which show that under certain circumstances stably parallelizable implies parallelizable. The purpose of this note is to give a more elementary proof of the parallelizability of the $V_{n,q}$ for $q > 2$.

2. Preliminaries and statement of the theorem. Let $S^{n-1}$ denote the unit sphere in $\mathbb{R}^n$. $V_{n,q}$ can then be regarded as the subspace of $S^{n-1} \times \cdots \times S^{n-1}$, $q$ times, consisting of all $(x_1, \cdots, x_q)$ in $S^{n-1} \times \cdots \times S^{n-1}$ with $x_i \perp x_j$ for $i \neq j$. As such, $V_{n,q}$ is a differentiable submanifold of $S^{n-1} \times \cdots \times S^{n-1}$.

**Theorem.** $V_{n,q}$ has a trivial normal bundle in $S^{n-1} \times \cdots \times S^{n-1}$. If $q > 2$, then $V_{n,q}$ is parallelizable.

For any bundle $\xi$, write $\xi_x$ for the fibre over a point $x$ in the base space. Write $\tau$ for the tangent bundle of $S^{n-1} \times \cdots \times S^{n-1}$, $\nu$ for the normal bundle of $V_{n,q}$ in $S^{n-1} \times \cdots \times S^{n-1}$. Then $\tau|V_{n,q} = \tau \oplus \nu$, where $\oplus$ denotes Whitney sum.

If $x = (x_1, \cdots, x_q) \in S^{n-1} \times \cdots \times S^{n-1}$, $\tau_x$ consists of all $(x; u_1, \cdots, u_q)$ where $u_i \in \mathbb{R}^n$, $u_i \perp x_i$, $1 \leq i \leq q$. The inner product of $(x; u_1, \cdots, u_q)$ and $(x; u'_1, \cdots, u'_q)$ in $\tau_x$ is $\sum_i u_i \cdot u'_i$. For $x \in V_{n,q}$, $\tau_x$ is the orthogonal complement of $\nu_x$ in $(\tau|V_{n,q})_x$.

3. The normal bundle $\nu$. If $x \in V_{n,q}$, then $(x; u_1, \cdots, u_q) \in (\tau|V_{n,q})_x$ is normal to $V_{n,q}$ if and only if

$$\lim_{x' \to x; x' \in V_{n,q}} \frac{(x - x') \cdot (u_1, \cdots, u_q)}{||x - x'||} = 0.$$  

For $1 \leq r < s \leq q$, let $\lambda_{r,s}$ denote the line subbundle of $\tau|V_{n,q}$ whose fibre $(\lambda_{r,s})_x$ is spanned by

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Theorem 2.2: Let $\lambda_{r,s}$ be a trivial bundle, and for $(r, s) \neq (r', s')$, $(\lambda_{r,s})_x$ is orthogonal to $(\lambda_{r',s'})_x$ for all $x \in V_{n,q}$. Thus, $\oplus_{1 \leq r < s \leq q} \lambda_{r,s}$ is a trivial $\frac{1}{2}q(q-1)$-plane subbundle of $\tilde{\tau} | V_{n,q}$.

4. The tangent bundle $\tau$. For $1 \leq i \leq q$, let $\alpha_i$ denote the subbundle of $\tilde{\tau} | V_{n,q}$ whose fibre $(\alpha_i)_x$ consists of all

\[ (x; 0, \ldots, 0, x_i, 0, \ldots, 0) \]

where $u \in \mathbb{R}^n$, $u \perp x_k$, $1 \leq k \leq q$. $\alpha_i$ is an $(n-q)$-plane bundle. Note that $(\alpha_i)_x$ is orthogonal to $v_x$ in $(\tilde{\tau} | V_{n,q})_x$, $1 \leq i \leq q$.

For $1 \leq r < s \leq q$, let $\beta_{r,s}$ denote the line subbundle of $\tilde{\tau} | V_{n,q}$ whose fibre $\beta_{r,s}$ is spanned by

\[ (x; 0, \ldots, 0, x_r, 0, \ldots, 0, -x_r, 0, \ldots, 0) \]

where $u \in \mathbb{R}^n$, $u \perp x_k$, $1 \leq k \leq q$. $\beta_{r,s}$ is a trivial line subbundle of $\tilde{\tau} | V_{n,q}$. Note that $(\beta_{r,s})_x$ is orthogonal to $v_x$ to all $(\alpha_i)_x$, and to $(\beta_{r',s'})_x$ for $(r, s) \neq (r', s')$. Hence $\oplus_{1 \leq i \leq q} \alpha_i$
\( \oplus (\oplus_{1 \leq r \leq q} \beta_{r,s}) \) is a subbundle of \( \tau \) of fibre dimension \( q(n-q) + \frac{1}{2}q(q-1) = q(n-1) - \frac{1}{2}q(q-1) = \dim V_{n,q}. \) Hence,

\[
\tau = \left( \oplus_{1 \leq i \leq q} \alpha_i \right) \oplus \left( \oplus_{1 \leq r \leq q} \beta_{r,s} \right) \cong \left( \oplus_{i} \alpha_i \right) \oplus \frac{1}{2} q(q-1).
\]

The trivial \( n \)-plane bundle \( V_{n,q} \times \mathbb{R}^n \) over \( V_{n,q} \) splits as the Whitney sum \( \alpha \oplus \gamma \) where \( \alpha \) consists of all \((x, u), u \in \mathbb{R}^n, u \perp x_k, 1 \leq k \leq q, \) and \( \gamma \) consists of all \((x, v), v \in \mathbb{R}^n, v \) in the span of \( x_1, \ldots, x_q. \) \( \gamma \) is a trivial \( q \)-plane bundle, having the \( q \) everywhere linearly-independent cross-sections \( s_i \) defined by \( s_i(x) = (x, x_i), 1 \leq i \leq q. \) Hence, \( \alpha \oplus q \cong \mathbb{R}^n. \) Hence, \( k\alpha \oplus l \) is trivial if \( l \geq q, \) and \( k \) is any positive integer.

Each of the \( \alpha_i \) above is equivalent to \( \alpha. \) Hence \( \tau \cong q\alpha \oplus \frac{1}{2} q(q-1). \) If \( q > 2, \frac{1}{2}q(q-1) \geq q, \) and so \( \tau \) is trivial.

**References**


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