REPRESENTATIONS OF LIE ALGEBRAS BY NORMAL OPERATORS

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Recently I. E. Segal [2] has proposed a study of the unitary representations of a complex semisimple Lie group G by studying "analytic" holomorphic representations of G by normal operators. To this end he proved that every unitary representation U of G may be written $U(g)=R(g)R(g^{-1})^*$ $(g\in G)$ where R is an analytic holomorphic representation of G by normal operators such that if $R(g_1)$ and $R(g_2)$ are defined then $R(g_1)$ commutes with $R(g_2)^*$.

Now a representation of a group by normal operators is a peculiar phenomenon because the product of two normal operators is usually not normal. In this paper we study this peculiarity in Lie algebraic terms. We achieve a decomposition of a representation of a semisimple Lie algebra by normal operators into the sum of two representations which commute with each other. One of these is by skew-adjoint operators, and the other is a representation (necessarily by normal operators) which commutes with its contragredient.

We begin by establishing our results in the following generalised setting: Let $\mathfrak{u}$ be a Lie algebra over some fixed field of characteristic other than 2. On $\mathfrak{u}$ we assume the existence of a linear mapping sending an element x of $\mathfrak{u}$ into x^* such that $(x^*)^*=x$ and $[x^*, y^*] = -[x, y]^*$. In other words our mapping is an anti-automorphism of order 2. If $\mathfrak{a}$ and $\mathfrak{b}$ are subsets of $\mathfrak{u}$ let $[\mathfrak{a}, \mathfrak{b}]=$ the linear span of $\{[x, y] \mid x\in\mathfrak{a}, y\in\mathfrak{b}\}$ and $\mathfrak{a}^*=\{x^* \mid x\in\mathfrak{a}\}$. We will say that an element x of $\mathfrak{u}$ is *nrml* if $[x, x^*]=0$ and that x is *skew* if $x^*=-x$.

LEMMA. *Let $\mathfrak{g}$ be a subalgebra of $\mathfrak{u}$ consisting of nrml elements. Then $\mathfrak{g}^+=\mathfrak{g}+\mathfrak{g}^*+[\mathfrak{g}, \mathfrak{g}^*]$ is a Lie algebra and $\mathfrak{i}=[\mathfrak{g}, \mathfrak{g}^*]$ is an ideal in $\mathfrak{g}^+$ consisting of skew elements.*

PROOF. For any x and y in $\mathfrak{g}$ we have

$$\begin{aligned} 0 &= [x + y, (x + y)^*] = [x, x^*] + [x, y^*] + [y, x^*] + [y, y^*] \\ &= [x, y^*] + [y, x^*]. \end{aligned}$$

But $[y, x^*]=-[y^*, x]^*$ so $[x, y^*]$ is skew. Since every element of $[\mathfrak{g}, \mathfrak{g}^*]$ is a linear combination of such elements we have at least shown that $[\mathfrak{g}, \mathfrak{g}^*]$ consists entirely of skew elements. To prove that $\mathfrak{i}=[\mathfrak{g}, \mathfrak{g}^*]$

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is an ideal, we have only to show that $[\mathfrak{g}, \mathfrak{i}] \subseteq \mathfrak{i}$. For then $[\mathfrak{g}^*, \mathfrak{i}] = [\mathfrak{g}, \mathfrak{i}]^* \subseteq \mathfrak{i}^* = \mathfrak{i}$ and

$$[\mathfrak{i}, \mathfrak{i}] = [[\mathfrak{g}, \mathfrak{g}^*], \mathfrak{i}] \subseteq [[\mathfrak{g}, \mathfrak{i}], \mathfrak{g}^*] + [\mathfrak{g}, [\mathfrak{g}^*, \mathfrak{i}]] \subseteq [\mathfrak{g}^*, \mathfrak{i}] + [\mathfrak{g}, \mathfrak{i}] \subseteq \mathfrak{i}.$$

So $\mathfrak{g}+\mathfrak{g}^*+\mathfrak{i}$ would be shown to be a Lie algebra and $\mathfrak{i}$ an ideal.

To see that $[\mathfrak{g}, \mathfrak{i}] \subseteq \mathfrak{i}$ we have only to show that for x, y, z in $\mathfrak{g}$, $[x, [y, z^*]]$ is in $\mathfrak{i}$. Now

$$[z, [y, z^*]] = [[z, y], z^*] + [y, [z, z^*]] = [[z, y], z^*] \in \mathfrak{i}.$$

So $[x+z, [y, (x+z)^*]]$ is in $\mathfrak{i}$. But

$$[x + z, [y, (x + z)^*]] = [x, [y, x^*]] + [x, [y, z^*]] + [z, [y, x^*]] + [z, [y, z^*]].$$

The first and last terms are in $\mathfrak{i}$, so $[x, [y, z^*]] + [z, [y, x^*]] \in \mathfrak{i}$. We claim that $[z, [y, x^*]] \equiv [x, [y, z^*]] \pmod{\mathfrak{i}}$. To see this note that

$$[y, x^*] = [x, y^*]^* = - [x, y^*] = [y^*, x].$$

So we have

$$\begin{aligned}[z, [y, x^*]] &= [z, [y^*, x]] = [[z, y^*], x] + [y^*, [z, x]] \\ &\equiv [[z, y^*], x] \pmod{\mathfrak{i}} \equiv [[z^*, v], x] \pmod{\mathfrak{i}} \\ &\equiv [x, [y, z^*]] \pmod{\mathfrak{i}}.\end{aligned}$$

And we finally obtain

$$2[x, [y, z^*]] \equiv ([x, [y, z^*]] + [z, [y, x^*]]) \pmod{\mathfrak{i}} \equiv 0 \pmod{\mathfrak{i}} \qquad \text{Q.E.D.}$$

We now assume that the ground field is of characteristic 0. Then the finite-dimensional representations of a semisimple Lie algebra are completely reducible. (See for example [**1**, Theorem 8, p. 79].)

THEOREM. *Let* $\mathfrak{g}$ *be a semisimple subalgebra of* $\mathfrak{u}$. *Then for* x *in* $\mathfrak{g}$ *we may uniquely write* $x = x_0 + x_1$ *where* x_0 *is nrml and* x_1 *is skew. Moreover, for any* y *in* $\mathfrak{g}$, *if we similarly write* $y = y_0 + y_1$ *then* $[x_0, y_1] = [y_0, x_1] = 0$ *and* $[x_0, (y_0)^*] = 0$. *Stated in other terms, the minimal self-adjoint subalgebra* $\mathfrak{g}^+$ *of* $\mathfrak{u}$ *containing* $\mathfrak{g}$ *is the Lie algebra direct sum of three ideals,* $\mathfrak{a}$, $\mathfrak{a}^*$, *and* $\mathfrak{i}$, *with* $\mathfrak{g} \subseteq \mathfrak{a} + \mathfrak{i}$ *and* $x^* = x$ *for* $x \in \mathfrak{i}$.

PROOF. We will apply the fact that the finite-dimensional representations of $\mathfrak{g}$ are completely reducible to the adjoint representation of $\mathfrak{g}$ on ideals in $\mathfrak{g}^+ = \mathfrak{g} + \mathfrak{g}^* + \mathfrak{i}$, $\mathfrak{i} = [\mathfrak{g}, \mathfrak{g}^*]$.

Let $\mathfrak{g}_0 = \{x \in \mathfrak{g}^+ \mid [x, \mathfrak{i}] = \{0\}\}$. We wish to make several observations about $\mathfrak{g}_0$. First, $\mathfrak{g}_0$ is an ideal in $\mathfrak{g}^+$. In fact it is the centralizer of the ideal $\mathfrak{i}$ in $\mathfrak{g}^+$ and is therefore an ideal.

Second, $\mathfrak{g}_0^* = \mathfrak{g}_0$. Indeed, if $x \in \mathfrak{g}_0$ then

$$[x^*, \mathfrak{i}] = [x, \mathfrak{i}^*]^* = [x, \mathfrak{i}]^* = \{0\}.$$

So $x^* \in \mathfrak{g}_0$. Since $x^{**} = x$, we have $x \in \mathfrak{g}_0$ if and only if $x^* \in \mathfrak{g}_0$.

Third, $\mathfrak{g}_0 + \mathfrak{i} = \mathfrak{g}^+$. To see this note that for any $x \in \mathfrak{g}^+$ such that $x^* = x$, and any $y \in \mathfrak{i}$, we have

$$-[x, y] = [x, y]^* = [y^*, x^*] = [-y, x] = [x, y].$$

So $[x, y] = 0$. Thus $[x, \mathfrak{i}] = \{0\}$; so $x \in \mathfrak{g}_0$. If we pick $z \in \mathfrak{g}$ and let $x = z + z^*$ then $z + z^* \in \mathfrak{g}_0$. For any $y \in \mathfrak{g}$ we have then that $[y, z] = [y, z+z^*] - [y, z^*]$. The first term is in $\mathfrak{g}_0$ since $\mathfrak{g}_0$ is an ideal; the second term is in $\mathfrak{i}$. So we have that $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}_0 + \mathfrak{i}$. Since $\mathfrak{g}$ is semisimple, $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}_0 + \mathfrak{i}$. Since $\mathfrak{g}_0^* = \mathfrak{g}_0$, we have that

$$\mathfrak{g}^* \subseteq (\mathfrak{g}_0 + \mathfrak{i})^* = (\mathfrak{g}_0)^* + \mathfrak{i}^* = \mathfrak{g}_0 + \mathfrak{i}.$$

So

$$\mathfrak{g}^+ = \mathfrak{g} + \mathfrak{g}^* + \mathfrak{i} \subseteq \mathfrak{g}_0 + \mathfrak{i}.$$

Fourth, $\mathfrak{g}_0 \cap \mathfrak{i} = \{0\}$. Let $\mathfrak{i}_0 = \mathfrak{g}_0 \cap \mathfrak{i}$. Then

$$[\mathfrak{g}_0, \mathfrak{i}_0] \subseteq [\mathfrak{g}_0, \mathfrak{i}] = \{0\}, \quad [\mathfrak{i}, \mathfrak{i}_0] \subseteq [\mathfrak{i}, \mathfrak{g}_0] = \{0\},$$
$$[\mathfrak{g}^+, \mathfrak{i}_0] \subseteq [\mathfrak{g}_0 + \mathfrak{i}, \mathfrak{i}_0] \subseteq [\mathfrak{g}_0, \mathfrak{i}_0] + [\mathfrak{i}, \mathfrak{i}_0] = \{0\}.$$

In particular, $[\mathfrak{g}, \mathfrak{i}_0] = \{0\}$. Since $\mathfrak{i}$ is completely reducible under the action of $\mathrm{ad}(\mathfrak{g})$, there is a linear compliment $\mathfrak{i}_1$ in $\mathfrak{i}$ to $\mathfrak{i}_0$ such that $[\mathfrak{g}, \mathfrak{i}_1] \subseteq \mathfrak{i}_1$. So

$$[\mathfrak{g}, \mathfrak{i}] = [\mathfrak{g}, \mathfrak{i}_0 + \mathfrak{i}_1] \subseteq [\mathfrak{g}, \mathfrak{i}_0] + [\mathfrak{g}, \mathfrak{i}_1] \subseteq [\mathfrak{g}, \mathfrak{i}_1] \subseteq \mathfrak{i}_1.$$

We may show however that $[\mathfrak{g}, \mathfrak{i}] = \mathfrak{i}$, proving $\mathfrak{i}_1 = \mathfrak{i}$ and $\mathfrak{i}_0 = \{0\}$. To see that $[\mathfrak{g}, \mathfrak{i}] = \mathfrak{i}$, consider the action of $\mathrm{ad}(\mathfrak{g})$ on $\mathfrak{g}^* + \mathfrak{i}$. Since $\mathfrak{g}^* + \mathfrak{i}$ is completely reducible under $\mathrm{ad}(\mathfrak{g})$, and $[\mathfrak{g}, \mathfrak{g}^* + \mathfrak{i}] \subseteq \mathfrak{i}$, we may find a complement $\mathfrak{i}^\perp$ in $\mathfrak{g}^* + \mathfrak{i}$ to $\mathfrak{i}$ such that $[\mathfrak{g}, \mathfrak{i}^\perp] \subseteq \mathfrak{i}^\perp$. But $[\mathfrak{g}, \mathfrak{i}^\perp] \subseteq \mathfrak{i}$ so $[\mathfrak{g}, \mathfrak{i}^\perp] \subseteq \mathfrak{i} \cap \mathfrak{i}^\perp = \{0\}$. Now

$$\mathfrak{i} = [\mathfrak{g}, \mathfrak{g}^*] \subseteq [\mathfrak{g}, \mathfrak{i}^\perp + \mathfrak{i}] = [\mathfrak{g}, \mathfrak{i}] + [\mathfrak{g}, \mathfrak{i}^\perp] = [\mathfrak{g}, \mathfrak{i}] \subseteq \mathfrak{i}.$$

So $\mathfrak{i} = [\mathfrak{g}, \mathfrak{i}]$. Consequently $\mathfrak{g}_0 \cap \mathfrak{i} = \{0\}$.

We have expressed $\mathfrak{g}^+$ as the direct sum of two ideals, $\mathfrak{g}_0$ and $\mathfrak{i}$. Thus for all $x, y \in \mathfrak{g}$ we may uniquely write $x = x_0 + x_1$ and $y = y_0 + y_1$ where $x_0, y_0 \in \mathfrak{g}_0$ and $x_1, y_1 \in \mathfrak{i}$. Clearly $[x_0, y_1] = [x_1, y_0] = 0$. Also $[x_0, (y_0)^*] = [(x - x_1), (y - y_1)^*] \equiv [x, y^*] \pmod{\mathfrak{i}} \equiv 0 \pmod{\mathfrak{i}}$. But $\mathfrak{g}_0^* = \mathfrak{g}_0$ so $[x_0, (y_0)^*] \subseteq \mathfrak{g}_0 \cap \mathfrak{i} = \{0\}$. Q.E.D.

As an application of this theorem let V be a complex vector space, not necessarily finite dimensional, with an inner product denoted

$\langle\ ,\ \rangle$, so that V may be regarded as a dense subset of a Hilbert space. (In applications to analytic representations of Lie groups, V would be the "analytic domain" of [2].) By "operator" on V we will mean a linear operator defined everywhere. An operator x on V has an adjoint if there is an operator y on V such that for all $v, v' \in V$, $\langle x(v), v'\rangle = \langle v, y(v')\rangle$. If y exists it is unique and we write $x^* = y$. Let $\mathfrak{u}$ be the set of operators on V having an adjoint. Then $\mathfrak{u}$ is a linear space and in fact a Lie algebra with the usual bracket. (The adjoint of $x+y$ is x^*+y^* and the adjoint of xy is y^*x^*.) $\mathfrak{u}$ is stable under the map $x \to x^*$ which is an anti-automorphism of order 2. We therefore have the following corollaries of the Lemma and Theorem:

COROLLARY 1. *Let $\mathfrak{g}$ be a complex Lie algebra and π a holomorphic representation of $\mathfrak{g}$ on V by nrml operators in $\mathfrak{u}$. Then π commutes with its contragredient.*

PROOF. That π is holomorphic means simply that it is complex linear. Identify $\mathfrak{g}$ with its image under π in $\mathfrak{u}$. By the Lemma we have that for all $x, y \in \mathfrak{g}$, $[x, y^*]$ is skew. Now ix is also in $\mathfrak{g}$, so $[ix, y^*] = i[x, y^*]$ is also skew. What we must show is that $[x, y^*] = 0$. But

$$\begin{aligned} i[x, y^*] &= -i(-[x, y^*]) = -i([x, y^*])^* \\ &= (i[x, y^*])^* = -i[x, y^*] = 0. \end{aligned}$$ Q.E.D.

COROLLARY 2. *Let $\mathfrak{g}$ be a real semisimple Lie algebra and π a representation of $\mathfrak{g}$ on V by nrml operators in $\mathfrak{u}$. Then there are representations π_0 and π_1 of $\mathfrak{g}$ on V by operators in $\mathfrak{u}$, such that π_0 commutes with its contragredient, π_1 is a representation by skew elements, π_0 commutes with π_1, and for all $x \in \mathfrak{g}$, $\pi(x) = \pi_0(x) + \pi_1(x)$.*

PROOF. Identify $\mathfrak{g}$ with its image under π in $\mathfrak{u}$ and apply the Theorem. Q.E.D.

A similar result holds over any field of characteristic 0 if we replace the Hermitian inner product on V by some nonsingular symmetric bilinear form.

Professor Segal pointed out the following application of Corollary 2:

COROLLARY 3. *Let $\mathfrak{g}$ be a real semisimple Lie algebra and π a finite-dimensional representation on a complex vector space by normal operators. Then π is a representation by skew-adjoint operators.*

PROOF. We apply Corollary 2 with V finite dimensional. Then $\mathfrak{u}$ is the Lie algebra of all complex linear operators, nrml means normal, and skew means skew-adjoint. For the decomposition $\pi = \pi_0 + \pi_1$ we must show that $\pi_0 = 0$. To do this extend π_0 to a holomorphic representation of the complexification of $\mathfrak{g}$ by complex linearity, again denoting this representation π_0. Let π_- be defined by $\pi_-(x) = \pi_0(x) - \pi_0(x)^*$ for all x in the complexification of $\mathfrak{g}$. Since the extended representation π_0 commutes with its contragredient, π_- is a representation by skew-adjoint operators of a complex semisimple Lie algebra. But it is well known that no such finite-dimensional representations exist. So for all x in $\mathfrak{g}$ we have $0 = \pi_-(x) = \pi_0(x) - \pi_0(x)^*$ or in other words, π_0 is a representation by self-adjoint operators. But since the bracket of two such operators is skew-adjoint and $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ we must have that $\pi_0 = 0$. Q.E.D.

To complete the picture of finite-dimensional representations by normal operators we remark that if $\mathfrak{g}$ is a real solvable Lie algebra and π a representation on a finite-dimensional complex vector space V by normal operators then $\pi([\mathfrak{g}, \mathfrak{g}]) = \{0\}$. Indeed by Lie's theorem we may find a basis of V with respect to which π is a representation by upper triangular matrices. The bracket of two such matrices is nilpotent so the image of $[\mathfrak{g}, \mathfrak{g}]$ under π is a set of nilpotent normal matrices; but the only matrix with both these properties is 0.

Finally, if $\mathfrak{g}$ is any real Lie algebra, write $\mathfrak{g} = \mathfrak{k} + \mathfrak{r}$ by the Levi decomposition, where $\mathfrak{k}$ is semisimple and $\mathfrak{r}$ is the maximal solvable ideal in $\mathfrak{g}$. Suppose π is a faithful finite-dimensional representation by normal operators on a complex vector space. Then by the above remark, $\mathfrak{r}$ is abelian; by Corollary 3, π restricted to $\mathfrak{k}$ is skew-adjoint. The set $[\mathfrak{k}, \mathfrak{r}]$ is stable under the action of $\operatorname{ad}(\mathfrak{k})$ on $\mathfrak{r}$ and consequently has a linear compliment in $\mathfrak{r}$ stable under $\operatorname{ad}(\mathfrak{k})$, namely the center of $\mathfrak{g}$. Now for $x \in \pi(\mathfrak{k})$, $y \in \pi(\mathfrak{r})$, we have $[x, y] = [x, (y - y^*)] + [x, y^*]$. The last term is skew-adjoint (by the Lemma) as is the first, so the operators of $\pi([\mathfrak{k}, \mathfrak{r}])$ are skew-adjoint. But a Lie algebra may have a finite-dimensional representation by skew-adjoint operators only if it is the direct sum of a compact semisimple Lie algebra and a central subalgebra. Applying this fact to $\mathfrak{k} + [\mathfrak{k}, \mathfrak{r}]$ we have $[\mathfrak{k}, [\mathfrak{k}, \mathfrak{r}]] = 0$. So all of $\mathfrak{r}$ commutes with $\mathfrak{k}$. Thus $\mathfrak{g}$ is the Lie algebra direct sum of a compact semisimple Lie algebra and an abelian Lie algebra. Thus the assumption that a finite-dimensional representation of a Lie algebra is by normal operators rather than skew-adjoint ones is no real increase of generality.

Bibliography

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